TD 1: Bounded operators

Notation: we use the notation $L^p(X) = L^p(X, \mathbb{C})$ for the Lebesgue space of complex valued functions, and $C^0(X) = C^0(X, \mathbb{C})$ for continuous functions. $\mathcal{L}(E, F)$ denotes the set of linear operators from E to F and $\mathcal{B}(E, F)$ the set of bounded operators. When $E = F$, we will just write $\mathcal{L}(E)$ or $\mathcal{B}(E)$. The space of power p summable sequences of complex number is denoted by ℓ^p.

Exercise 1 – ***Operator norm and adjoint***. Let E, F and G be Banach spaces, $A \in \mathcal{L}(E, F)$ and $B \in \mathcal{L}(F, G)$.

1. Assume A and B are bounded. Prove that $BA \in \mathcal{B}(E, G)$ and $\|BA\| \leq \|B\| \, \|A\|$.
2. Prove that A is bounded if and only if it is continuous.
3. Using the Hahn–Banach theorem, prove that for any $x \in E$,
$$\|x\|_E = \sup_{\substack{y \in E' \\ \|y\| \leq 1}} |\langle x, y \rangle| \,.$$
4. If $A \in \mathcal{B}(E, F)$, prove that one can define its adjoint $A^* \in \mathcal{B}(F', E')$ by
$$\langle x, A^* y \rangle := \langle Ax, y \rangle \,.$$

Exercise 2 – ***Almost invertible operators***. Give explicit examples of bounded operators A and B on ℓ^2 such that $AB = \mathrm{Id}_{\ell^2}$ and BA is the projection onto a closed infinite-dimensional subspace of infinite codimension.

Exercise 3 – ***Weighted shift***. We denote by $(e_n)_{n \in \mathbb{N}^*}$ the standard Hilbert basis of ℓ^2 and $e_0 = 0 \in \ell^2$. Let $a = (a_n)_{n \in \mathbb{N}} \in \ell^\infty$.

1. Show that there is a unique bounded operator A on ℓ^2 such that for any $n \in \mathbb{N}^*$, $Ae_n = a_n \, e_{n+1}$ and a unique bounded operator B on ℓ^2 such that for any $n \in \mathbb{N}^*$, $Be_n = a_n \, e_{n-1}$. Compute their operator norms and adjoint.
2. Find the eigenvalues of A and B, that is for $\lambda \in \mathbb{R}$, solve the equations $Au = \lambda u$ and $Bu = \lambda u$.
3. For the two operators above with $a_n = 1$ for all $n \in \mathbb{N}$, compute AB and BA and deduce that A is injective but not surjective, B is surjective but not injective.
4. Assuming that $a_n \to 0$ as $n \to \infty$, show that $\lim_{n \to \infty} \|A^n\|^{1/n} = 0$.

Exercise 4 – ***Multiplication operators***. To any function $u \in L^\infty(\mathbb{R})$, we associate the operator M_u defined for $\psi \in L^2(\mathbb{R})$ by
$$M_u \psi(x) = u(x) \, \psi(x) \,.$$

1. Prove that u is a bounded operator and compute its operator norm and its adjoint.
2. Assume u is a real-valued strictly increasing function. Then prove that for any $\lambda \in \mathbb{C}$, the equation $M_u \psi = \lambda \, \psi$ has no solution.

Exercise 5 – ***Differential operator***. Recall that for $n \in \mathbb{N}$, one can define $H^2(\mathbb{R}^d)$ as the space of functions such that the norm
$$\|u\|_{H^2} := \left(\|u\|_{L^2}^2 + \left\| \nabla^2 u \right\|_{L^2}^2 \right)^{1/2}$$
is finite. Here $\nabla^2 u$ denotes the Hessian of u.

1. Prove that $A := 1 - \Delta \in \mathcal{B}(H^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$. Find its operator norm and its adjoint.
2. Prove that A is invertible.

Exercise 6 – ***Integral operators***. Let $K \subseteq \mathbb{R}^d$ be compact and $a \in C^0(K^2)$ and define $A \in \mathcal{L}(L^1(K), C^0(K))$ for any $\psi \in L^1(K)$ by
$$A\psi(x) = \int_K a(x, y) \, \psi(y) \, \mathrm{d}y.$$

1. Prove that A is well-defined, $A \in \mathcal{B}(L^1(K), C^0(K))$.
2. Prove that the image of a bounded set of $C^0(K)$ by A is a compact set of $C^0(K)$.
3. Prove that if $a \in L^2(K)$ with K possibly unbounded, then the above expression also defines an operator $A \in \mathcal{B}(L^2(K))$.

TD 2: Banach algebra and spectrum

Exercise 1 – ***Banach Algebra***.

1. If E is a Banach space, prove that $\mathcal{B}(E)$ is a Banach algebra.
2. Let A be a $\mathbb{K}$-algebra that is also a Banach space for some norm $N : A \to \mathbb{R}_+$. Prove that if the multiplication is continuous, then there exists a norm $\|\cdot\|$ such that A is a Banach algebra.
3. Let $\mathcal{F}(L^1) = \{ f \in \mathcal{D}'(\mathbb{R}) : \widehat{f} \in L^1(\mathbb{R}) \}$. Is $\mathcal{F}(L^1)$ a Banach algebra for the usual multiplication of functions? Same question with $\mathcal{F}(\mathcal{M})$ where $\mathcal{M}$ denotes the set of finite measures on $\mathbb{R}$.

Exercise 2 – ***Inversion and connected sets***. Let A be a Banach algebra.

1. Prove that $A^\times$ is an open set and a topological group.
2. Show that for every element $x \in A$ satisfying $\|x\| < 1$, there is a continuous function $f : [0,1] \to A^\times$ such that $f(0) = 1$ and $f(1) = (1-x)^{-1}$.
3. Show that for every element $x \in A^\times$, there is an $\varepsilon > 0$ with the following property: for every element $y \in A^\times$ satisfying $\|y - x\| \leq \varepsilon$, there is an arc in $A^\times$ connecting y to x.
4. Prove that an open subgroup of $A^\times$ is always closed.
5. Let G be the set of all finite products of elements of $A^\times$ of the form $1 - x$ or $(1-x)^{-1}$, where $x \in A$ satisfies $\|x\| < 1$. Show that G is the connected component of 1 in $A^\times$.

Exercise 3 – ***Spectrum of functions***. Let X be a compact Hausdorff space and let $A = C^0(X)$ be the Banach algebra of all complex-valued continuous functions on X. If $f \in C^0(X)$, what is its spectrum?

Exercise 4 – ***Multiplication operator***. Let $\Omega \subseteq \mathbb{R}$ be a compact set and M_x be the operator defined for functions $\psi \in L^2(\Omega)$ by

$$M_x\psi(x) = x\,\psi(x)\,.$$

Find the spectrum of M_x.

Exercise 5 – ***Volterra equation of the second kind***. Let $k \in C^0([0,1]^2)$ and define $K \in \mathcal{B}(C^0([0,1]))$ for any $f \in C^0([0,1])$ by

$$Kf(x) = \int_0^x k(x,y)\,f(y)\,\mathrm{d}y\,.$$

1. Prove that for any $n \in \mathbb{N}$, $K^n \in \mathcal{B}(C^0(\Omega))$ and there exists a constant $c > 0$ such that for any $n \in \mathbb{N}$,

$$\|K^n\| \leq \frac{c^n}{n!}\,.$$

2. Show that for every complex number $\lambda \neq 0$ and every $g \in C^0([0,1])$, the equation $Kf = \lambda f + g$ has a unique solution $f \in C^0([0,1])$.

Exercise 6 – ***Differential operator***. What is the spectrum of the operator $(1-\Delta)^{-1}$ seen as a bounded operator on $L^2(\mathbb{R}^d)$?

TD 3: Spectral radius and decomposition of the spectrum

Exercise 1 – ***Volterra equation of the second kind***. Let $k \in C^0([0,1]^2)$ and define $K \in \mathcal{B}(C^0([0,1]))$ for any $f \in C^0([0,1])$ by

$$Kf(x) = \int_0^x k(x,y)\, f(y)\, \mathrm{d}y\,.$$

1. Prove that for any $n \in \mathbb{N}$, $K^n \in \mathcal{B}(C^0(\Omega))$ and there exists a constant $c > 0$ such that for any $n \in \mathbb{N}$,

$$\|K^n\| \leq \frac{c^n}{n!}\,.$$

2. Deduce that the spectral radius is different from the operator norm.

3. Show that for every complex number $\lambda \neq 0$ and every $g \in C^0([0,1])$, the equation $Kf = \lambda f + g$ has a unique solution $f \in C^0([0,1])$.

Exercise 2 – ***Spectral radius inequalities***. Let A be a complex Banach algebra and $a \in A$.

1. Prove that

$$r(a) \leq \inf_{b \in A^\times} \|b^{-1} a\, b\|\,.$$

2. Assume a and b have their spectral radius equal to their operator norm. Prove that

$$r(a+b) \leq r(a) + r(b).$$

Exercise 3 – ***Nilpotent operators***. Let A be a complex Banach algebra and a and b be two commuting nilpotent. Prove that $a + b$ is nilpotent. Is it still true if a and b do not commute?

Exercise 4 – ***Shift operator***. Let $\tau_l, \tau_r \in \mathcal{B}(\ell^2)$ be defined by

$$\tau_l(x_1, \ldots, x_n, \ldots) = (x_2, \ldots, x_n, \ldots)$$
$$\tau_r(x_1, \ldots, x_n, \ldots) = (0, x_1, x_2, \ldots, x_n, \ldots)\,.$$

1. What is their adjoint and spectral radius?

2. Compute the different parts of the spectrum σ_p, σ_a and σ_r of τ_l.

3. Compute the different parts of the spectrum σ_p, σ_a and σ_r of τ_r.

Exercise 5 – ***Multiplication operator***. Let (X, μ) be a probability space, $f \in L^\infty(X, \mu)$ and M_f be the operator defined for functions $\psi \in L^2(X, \mu)$ by

$$M_f \psi(x) = f(x)\, \psi(x)\,.$$

Prove that

- $\sigma_p(M_f) = \{\, z \in \mathbb{C} : \mu(f^{-1}(\{\, z \,\})) > 0 \,\}$
- $\sigma_a(M_f) = \{\, z \in \mathbb{C} : \forall \varepsilon > 0, \mu(f^{-1}(\{\, B_\varepsilon(z) \,\})) > 0 \,\}$
- $\sigma_r(M_f) = \emptyset$.

Exercise 6 – ***Differential operator***. What are the different parts of the spectrum of the operator $(1 - \Delta)^{-1}$ seen as a bounded operator on $L^2(\mathbb{R}^d)$?

TD 4: Normal operators, adjoints, and inequalities

We denote by H a complex Hilbert space.

Exercise 1 – ***Adjoint***. Let $*$ denote the adjoint operation on $\mathcal{B}(H)$.

1. Prove that $*$ is an involution of the complex Banach algebra $\mathcal{B}(H)$ (that is for any $A, B \in \mathcal{B}(H)$, $(AB)^* = B^*A^*$, $A^{**} = A$, and $(aA + bB)^* = \overline{a}A^* + \overline{b}B^*$ for any $(a, b) \in \mathbb{C}^2$) such that for any $A \in \mathcal{B}(A)$, invertible $(A^{-1})^* = (A^*)^{-1}$.
2. Prove that $*$ is continuous for the operator norm topology and the *weak operator topology* (that is $\forall x \in H$, $A_n x \to Ax$ weakly implies $A_n^* x \to A^* x$ weakly) but not for the *strong operator topology* (that is $\forall x \in H$, $A_n x \to Ax$ strongly does not imply $A_n^* x \to A^* x$ strongly).
3. Give an example of a normal operator different from 1 and a non-normal operator.

Exercise 2 – ***Quadratic form***. To every operator $A \in \mathcal{B}(H)$, one can associate a quadratic form $q_A : H \to \mathbb{C}$ defined by $q_A(x) = \langle Ax, x\rangle$. The numerical radius of A is defined by

$$w(A) = \sup_{x \in H, |x|=1} |q_A(x)| .$$

1. Show that A is self-adjoint if and only if q_A is real-valued.
2. Show that $w(A) \leq \|A\| \leq 2\, w(A)$.
3. Show that $q_A = q_B$ if and only if $A = B$.

Exercise 3 – ***Spectral Radius of normal operators***. Let $A \in \mathcal{B}(H)$.

1. Prove that $\|A^*A\| = \|A\|^2$.
2. Prove that if A is normal, then $\|A\| = r(A)$.

Exercise 4 – ***Square root***. Let $A \in \mathcal{B}(H)$ verify $A \geq 0$.

1. Prove that there exists an operator $B \in \mathcal{B}(H)$ such that $B \geq 0$, $B^2 = A$ and B commutes with every operator that commutes with A.
2. Prove that there exists a unique operator $B \in \mathcal{B}(H)$ such that $B \geq 0$, $B^2 = A$.

We denote by $\sqrt{A}$ the operator B defined in this way.

Exercise 5 – ***Inequalities for normal operators***. If $A \in \mathcal{B}(H)$, we define $|A| := \sqrt{A^*A}$. Let A and B be two normal operators on $\mathcal{B}(H)$.

1. Prove that for any $(m, n, p, q) \in \mathbb{N}^4$, $\|A^{m+n}B^{p+q}\| \leq \||A|^m |B|^p\| \, \||A|^n |B|^q\|$.
2. Prove that for any integers $k \leq n$, $\|A^k B^k\| \leq \||A|^n |B|^n\|^{k/n}$. *Hint: start with the case where k and n are powers of* 2.
3. If A is invertible and $0 \leq A \leq B$ (meaning $B - A \geq 0$) prove that $\sqrt{A} \leq \sqrt{B}$.
4. Remove the hypothesis of invertibility of A.

TD 5: Compact operators

Let E, F and G be Banach spaces and H be an Hilbert space. We denote by $\mathcal{K}(E,F)$ the space of compact operators from E to F.

Exercise 1 – ***Properties of compact operators.***

1. Let $A \in \mathcal{L}(E,F)$. Prove that if A compact, then A is bounded, and that the converse is true if E or F is finite dimensional.
2. Let $A \in \mathcal{B}(F,G)$ and $B \in \mathcal{B}(E,F)$. Prove that AB is compact if A or B is compact. Is the converse true?
3. Prove that if $K \in \mathcal{K}(E,F)$, then it maps weakly converging sequences to strongly convergent ones, that is if $(x_n)_{n\in\mathbb{N}}$ is a sequence converging weakly in E, then Kx_n converges strongly in F.
4. If E is reflexive, prove that $K \in \mathcal{K}(E,F)$ iff it maps weakly converging sequences to strongly convergent ones.
5. Let H be an infinite dimensional Hilbert space, and $K \in \mathcal{B}(H)$ be a compact operator. Show that for any orthonormal family $(e_n)_{n\in\mathbb{N}}$ of vectors of H,
$$\lim_{n\to\infty} \|Ke_n\| = 0\,.$$
6. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H, $Q_n := 1 - P_n$ where P_n is the orthogonal projection on $\mathrm{span}(e_1,\dots,e_n)$ and $A \in \mathcal{B}(H)$. Prove that
$$A \text{ is compact} \iff \|Q_n\,A\,Q_n\| \underset{n\to\infty}{\to} 0\,.$$

Exercise 2 – ***Hilbert–Schmidt integral operators.*** Let $H = L^2(\mathbb{R}^d,\mathbb{C})$ and $k \in L^2(\mathbb{R}^{2d},\mathbb{C})$ and define $K \in \mathcal{B}(L^2)$ by
$$\forall \psi \in H,\ K\psi(x) = \int_{\mathbb{R}^d} k(x,y)\,\psi(y)\,\mathrm{d}y\,.$$
The function k is called the integral kernel of K.

1. Prove that $\|K\| \le \|k\|_{L^2}$ and that the adjoint of K is an integral operator with integral kernel $\overline{k(x,y)}$.
2. Prove that K is a compact operator. *Hint: use the fact that $L^2(\mathbb{R}^d,\mathbb{C})$ is a separable Hilbert space and the compactness of finite rank operators.*

Exercise 3 – Let $A = u(x)\,(1-\Delta)^{-1} \in \mathcal{L}(L^2(\mathbb{R}))$ with $u \in L^2(\mathbb{R})$. Prove that A is compact.

Exercise 4 – Let $A \in \mathcal{B}(E,F)$ and B_E denote the unit ball of E.

1. Prove that if E is reflexive, then $A(B_E)$ is closed.
2. Prove that if E is reflexive and A is compact, then $A(B_E)$ is compact.
3. Let $E = F = C^0[0,1]$ and $Au(x) = \int_0^x u(t)\,\mathrm{d}t$. Prove that A is compact but $A(B_E)$ is not closed.

Exercise 5 – Let $1 \le p \le q \le \infty$ and $\Omega \subset \mathbb{R}^d$ be a bounded open set. Prove that the injection $L^q(\Omega) \to L^p(\Omega)$ is bounded but not compact. *Hint: strong oscillations ...*

TD 6 - Compact and Hilbert–Schmidt operators

Let H be a complex Hilbert space. We denote by $\mathcal{L}^2(H)$ the set of Hilbert–Schmidt operators on H.

Exercise 1 – Prove that the space $\mathcal{L}^2(H)$ is a Hilbert space.

Exercise 2 – ***Hilbert–Schmidt integral operators.*** Let $H = L^2(\mathbb{R}^d, \mathbb{C})$ and $k \in L^2(\mathbb{R}^{2d}, \mathbb{C})$ and define $K \in \mathcal{B}(H)$ by

$$\forall \psi \in H,\ K\psi(x) = \int_{\mathbb{R}^d} k(x,y)\, \psi(y)\, \mathrm{d}y\,.$$

The function k is called the integral kernel of K.

1. Prove that $\|K\| \le \|k\|_{L^2}$ and that the adjoint of K is an integral operator with integral kernel $\overline{k(x,y)}$.
2. Prove that $K \in \mathcal{L}^2(H)$.
3. Conversely, prove that if $A \in \mathcal{L}^2(H)$ then A is a Hilbert–Schmidt integral operator.

Exercise 3 – ***Your favorite operator.*** Let $A = u(x)\,(1-\Delta)^{-1} \in \mathcal{L}(H)$ with $u \in L^2(\mathbb{R}^d)$ and $H = L^2(\mathbb{R}^d, \mathbb{C})$.

1. If $d = 1$, prove that A is a Hilbert–Schmidt operator.
2. If $d \ge 1$ and $u \in (L^2 \cap L^\infty)(\mathbb{R}^d)$, prove that $A \in \mathcal{K}(H)$.

Exercise 4 – Let $T \in \mathcal{B}(H)$.

1. Prove that T is compact iff T^*T is compact.
2. Let $n \in \mathbb{N}^*$ and assume T is normal. Prove that T is compact iff T^n is compact. Is is still true if T is not normal?

Exercise 5 – Let $B \in \mathcal{L}^2(H)$ be self-adjoint and $A \in \mathcal{B}(H)$.

1. Prove that
$$\left\| |AB|^2 \right\|_2 \le \left\| |A|^2\, B^2 \right\|_2 .$$
2. Prove that for any $n \in \mathbb{N}$ which is a power of 2,
$$\| |AB|^n \|_2 \le \| |A|^n\, B^n \|_2 .$$

TD 6 - Compact and trace class operators

Let H be a complex Hilbert space. We denote by $\mathcal{L}^2(H)$ the set of Hilbert–Schmidt operators on H.

Exercise 1 – Let $T \in \mathcal{B}(H)$.

1. Prove that T is compact iff T^*T is compact.
2. Let $n \in \mathbb{N}^*$ and assume T is normal. Prove that T is compact iff T^n is compact. Is is still true if T is not normal?

Exercise 2 – ***Trace of positive operators.*** If $(e_n)_{n\in\mathbb{N}}$ is an Hilbert basis of H and $A \in \mathcal{B}(H)$ is a positive operator, one can define its trace $\mathrm{Tr}(A) \in [0, +\infty]$ as

$$\mathrm{Tr}(A) := \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle . \tag{1}$$

1. Prove that the above definition is independent of the basis.
2. Prove that for any $T \in \mathcal{B}(H)$, $\mathrm{Tr}(T^*T) = \mathrm{Tr}(TT^*)$ and this quantity is finite iff $T \in \mathcal{L}^2(H)$.

Exercise 3 – ***Cyclicity of the trace.*** If $A \in \mathcal{B}(H)$, then we define the set of trace class operators as

$$\mathcal{L}^1(H) = \mathrm{span}\,\{\, A \in \mathcal{B}(H) : A \geq 0 \text{ and } \mathrm{Tr}(A) < \infty \,\} .$$

1. Prove that any $A \in \mathcal{L}^1(H)$ can be written as a linear combination of four positive operators with finite trace.
2. Prove that if $A \in \mathcal{L}^1(H)$, then the series (1) makes sense and is absolutely convergent. It defines the trace on $\mathcal{L}^1(H)$.
3. Let $A, B \in \mathcal{L}^2(H)$. Prove that $AB \in \mathcal{L}^1(H)$ and $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.
4. Let $A \in \mathcal{K}(H)$ be normal and $B \in \mathcal{B}(H)$. Prove that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ remains true if $AB \in \mathcal{L}^1(H)$. Prove that this is in particular the case if $A \in \mathcal{L}^1(H)$.
5. If $A, B \in \mathcal{B}(H)$, is it always true that $\mathrm{Tr}([A, B]) = 0$?

Exercise 4 – ***Trace of normal operators.*** Let $A \in \mathcal{K}(H)$ be normal.

1. Prove that $A \in \mathcal{L}^2(H)$ iff the sequence of its eigenvalues $(\lambda_j(A))_{j\in\mathbb{N}}$ (counted with multiplicity) is in ℓ^2, and that in this case
$$\|A\|_2^2 = \mathrm{Tr}\Big(|A|^2\Big) = \sum_{j\in\mathbb{N}} |\lambda_j|^2 .$$
2. Prove that $A \in \mathcal{L}^1(H)$ iff $(\lambda_j(A))_{j\in\mathbb{N}} \in \ell^1$, and that in this case
$$\mathrm{Tr}(A) = \sum_{j\in\mathbb{N}} \lambda_j(A) .$$
3. Let $A \in \mathcal{L}^1(L^2(\mathbb{R}^d))$ be a normal integral operator with kernel $A(x, y) \in C^0(\mathbb{R}^{2d})$. Prove that $A(x, x) \in L^1(\mathbb{R}^d)$ and
$$\mathrm{Tr}(A) = \int_{\mathbb{R}^d} A(x, x)\, \mathrm{d}x .$$
Conversely, if $A(x, y) \in C^0(\mathbb{R}^{2d})$ is such that $A \geq 0$ and $A(x, x) \in L^1(\mathbb{R}^d)$, prove that $A \in \mathcal{L}^1(L^2(\mathbb{R}^d))$.

TD 8 - Polar decomposition and functional Calculus

Let H be a complex Hilbert space. We denote by $\mathcal{L}^2(H)$ the set of Hilbert–Schmidt operators on H and by $\mathcal{L}^1(H)$ the set of trace class operators.

Exercise 1 – ***Polar decomposition.*** Recall that if $A \in \mathcal{B}(H)$, then one can define $|A| := \sqrt{A^* A}$.

1. Prove that it is not always possible to write $A = U\,|A|$ for some unitary operator U.
2. Prove that there is a unique operator $U \in \mathcal{B}(H)$ such that $A = U\,|A|$ and $\ker U = \ker A$, and that it is an isometry from $(\ker U)^\perp$ to $\operatorname{ran} U$. *Define U on $\operatorname{ran}|A|$ first.*
3. Prove that U^* is an isometry from $\operatorname{ran} U$ to $(\ker U)^\perp$ and $|A| = U^* A$.

Exercise 2 – ***Trace class operators.***

1. Prove that
$$\mathcal{L}^1(H) = \{\, A \in \mathcal{B}(H) : \operatorname{Tr}(|A|) < \infty \,\}\,.$$
2. Prove that if $A \in \mathcal{B}(H)$ and $B \in \mathcal{L}^1(H)$, then
$$|\operatorname{Tr}(AB)| \le \|A\| \operatorname{Tr}(|B|)\,.$$
3. Prove that $\|A\|_{\mathcal{L}^1} := \operatorname{Tr}(|A|)$ is a norm on $\mathcal{L}^1$.

Exercise 3 – ***Compact operators and functional calculus.*** Let $A \in \mathcal{K}(H)$ be self-adjoint.

1. Prove that if $f \in C^0(\sigma(A))$ is such that $f(0) = 0$, then $f(A)$ is compact.
2. Prove that in this case
$$f(A) = \sum_{\lambda \in \sigma(A)} f(\lambda)\, P_\lambda\,.$$

Exercise 4 – ***Singular value decomposition.***

1. Prove that for any compact operator A, there exists a non-increasing sequence of positive numbers $(\mu_j(A))_{j\in\mathbb{N}}$ and two orthonormal sets $(\phi_j)_{j\in\mathbb{N}}$ and $(\psi_j)_{j\in\mathbb{N}}$ such that
$$A = \sum_{j\in\mathbb{N}} \mu_j(A)\, \langle \phi_j, \cdot\rangle\, \psi_j\,.$$
2. Prove that the $\mu_j(A)$ are unique and deduce that $\mu_j(A) = \mu_j(A^*)$.
3. if A is a compact operator, prove that
$$\operatorname{Tr}(|A|^p) = \sum_{j\in\mathbb{N}} \mu_j(A)^p\,.$$

Exercise 5 – Let A, B be two positive operators such that A^p and B^q are trace class, with $1 < q \le 2 \le p < \infty$ and $p = q'$.

1. Show that if ψ_n is an orthonormal basis of eigenvectors of B associated to the eigenvalues λ_n, then
$$\langle |AB|\, \psi_n, \psi_n\rangle \le \lambda_n\, \langle A^p \psi_n, \psi_n\rangle^{1/p}\,.$$
2. Prove that
$$\operatorname{Tr}(|AB|) \le \operatorname{Tr}(A^p)^{1/p} \operatorname{Tr}(B^q)^{1/q}\,.$$

TD 9 - Functional calculus

Let H be a complex Hilbert space.

Exercise 1 – ***Compact operators and functional calculus.*** Let $A \in \mathcal{K}(H)$ be self-adjoint.

1. Prove that if $f \in C^0(\sigma(A))$ is such that $f(0) = 0$, then $f(A)$ is compact.
2. Prove that in this case
$$f(A) = \sum_{\lambda \in \sigma(A)} f(\lambda)\, P_\lambda\,.$$

Exercise 2 – ***Decomposition of compact operators.***

1. Prove that for any compact operator A, there exists a non-increasing sequence of positive numbers $(\mu_j(A))_{j\in\mathbb{N}}$ and two orthonormal sets $(\phi_j)_{j\in\mathbb{N}}$ and $(\psi_j)_{j\in\mathbb{N}}$ such that
$$A = \sum_{j\in\mathbb{N}} \mu_j(A)\, \langle \phi_j, \cdot \rangle\, \psi_j\,.$$
2. Prove that the $\mu_j(A)$ are unique and deduce that $\mu_j(A) = \mu_j(A^*)$.
3. if A is a compact operator, prove that
$$\mathrm{Tr}(|A|^p) = \sum_{j\in\mathbb{N}} \mu_j(A)^p.$$

Exercise 3 – ***Functional calculus of bounded operators.***

1. Prove that two bounded normal operators commute iff their spectral projections commute.
2. Let $f \in C^0(\mathbb{C})$ and $(A_n)_{n\in\mathbb{N}}$ be a sequence of normal bounded operators that converges in norm to an operator A. Show that $f(A_n)$ converges in norm to $f(A)$.

Exercise 4 – ***Schur Lemma.*** Let $S \subseteq \mathcal{B}(H)$ be such that $S^* = S$. We want to prove that the two following assertions are equivalent.

(i) The only closed subspaces invariants by S are $\{\,0\,\}$ and H.

(ii) If $A \in \mathcal{B}(H)$ is such that $\forall B \in S, AB = BA$, then $A \in \mathbb{C}\,\mathrm{Id}$.

1. Prove that (ii) implies (i).
2. Assume (i). Prove by contradiction that (ii) holds if A is self-adjoint.
3. Prove that (i) and (ii) are equivalent.

Exercise 5 – ***Von Neumann Ergodic Theorem.*** Let U be a unitary operator on H (i.e. a normal isometry) and P be the orthogonal projection on $\ker(1-U)$. Prove that for any $\psi \in H$,
$$\frac{1}{n}\sum_{k=0}^{n-1} U^k\psi \underset{n\to\infty}{\to} P\psi\,.$$
Let (Ω,μ) a probability space and T a measure preserving bijection on Ω such that for any $E \subseteq \Omega$, $\mu(E \,\triangle\, T(E)) = 0 \implies \mu(E) \in \{\,0,1\,\}$. Then prove that for any $f \in L^2(\Omega)$,
$$\frac{1}{n}\sum_{k=0}^{n-1} f\circ T^k \underset{n\to\infty}{\to} \int f(x)\,\mu(\mathrm{d}x) \quad \text{in } L^2(\Omega)\,.$$

TD 10 - Functional calculus and Gelfand transform

Let H be a complex Hilbert space.

Exercise 1 – ***Schur Lemma.*** Let $S \subseteq \mathcal{B}(H)$ be such that $S^* = S$. We want to prove that the two following assertions are equivalent.

(i) The only closed subspaces invariants by S are $\{ 0 \}$ and H.

(ii) If $A \in \mathcal{B}(H)$ is such that $\forall B \in S, AB = BA$, then $A \in \mathbb{C}\,\mathrm{Id}$.

1. Prove that (ii) implies (i).
2. Assume (i). Prove by contradiction that (ii) holds if A is self-adjoint.
3. Prove that (i) and (ii) are equivalent.

Exercise 2 – ***Von Neumann Ergodic Theorem.*** Let U be a unitary operator on H (i.e. a normal isometry) and P be the orthogonal projection on $\ker(1-U)$. Prove that for any $\psi \in H$,

$$\frac{1}{n}\sum_{k=0}^{n-1} U^k \psi \underset{n\to\infty}{\to} P\psi\,.$$

Let (Ω, μ) a probability space and T a measure preserving bijection on Ω such that for any $E \subseteq \Omega$, $\mu(E \triangle T(E)) = 0 \implies \mu(E) \in \{ 0, 1 \}$. Then prove that for any $f \in L^2(\Omega)$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \underset{n\to\infty}{\to} \int f(x)\,\mu(\mathrm{d}x) \quad \text{in } L^2(\Omega)\,.$$

Exercise 3 – ***Wiener Algebra.*** Let $\mathcal{A} = \ell^1(\mathbb{Z})$ with multiplication given by the convolution, that is $(a * b)_n = \sum_{k\in\mathbb{Z}} a_k\, b_{n-k}$. For any $n \in \mathbb{Z}$, we denote by δ_n the sequence such that for $k \in \mathbb{Z}$, $(\delta_n)_k = \delta_{n,k}$ is 1 if $k = n$ and 0 else.

1. Prove that $\mathcal{A}$ is a Banach algebra. What is its unit sequence? If $n, m \in \mathbb{Z}$, what is the action of the operator $\delta_n * \cdot$, and what is $\delta_n * \delta_m$?
2. Prove that for any $\lambda \in \mathcal{U} := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$,
$$\omega_\lambda := \left(a \mapsto \sum_{n\in\mathbb{Z}} a_n\, \lambda^n \right) \in \sigma(\mathcal{A})\,.$$
3. Conversely, prove that if $\omega \in \sigma(A)$, then it can be written in the above form.
4. Deduce that $\sigma(A)$ is homeomorphic to $\mathcal{U}$ and that one can identify the Gelfand transform with a discrete Fourier transform.
5. Let $\mathcal{W}$ be the set of continuous 2π-periodic functions whose Fourier series converges absolutely. Prove that if f has no zeros, then $1/f \in \mathcal{W}$.

Exercise 4 – Let $\mathcal{A} = C^0(X)$ for some compact Hausdorff space X.

1. Prove that for any proper ideal $\mathcal{I} \subset \mathcal{A}$ there exists $x \in X$ such that $\forall f \in \mathcal{I}, f(x) = 0$.
2. Prove that $\mathcal{I}$ is maximal iff the above property is true for exactly one $x \in X$.
3. Show that there is a bijection between the set of compact subsets of X and the set of closed ideals of $C^0(X)$.

TD 11 - Gelfand transform and Commutant

Let H be a complex Hilbert space.

Exercise 1 – ***Wiener Algebra.*** Let $\mathcal{A} = \ell^1(\mathbb{Z})$ with multiplication given by the convolution, that is $(a * b)_n = \sum_{k\in\mathbb{Z}} a_k\, b_{n-k}$. For any $n \in \mathbb{Z}$, we denote by δ_n the sequence such that for $k \in \mathbb{Z}$, $(\delta_n)_k = \delta_{n,k}$ is 1 if $k = n$ and 0 else.

1. Prove that $\mathcal{A}$ is a Banach algebra. What is its unit sequence? If $n, m \in \mathbb{Z}$, what is the action of the operator $\delta_n * \cdot$, and what is $\delta_n * \delta_m$?
2. Prove that for any $\lambda \in \mathcal{U} := \{\, \lambda \in \mathbb{C} : |\lambda| = 1 \,\}$,
$$\omega_\lambda := \left(a \mapsto \sum_{n\in\mathbb{Z}} a_n\, \lambda^n \right) \in \sigma(\mathcal{A})\,.$$
3. Conversely, prove that if $\omega \in \sigma(\mathcal{A})$, then it can be written in the above form.
4. Deduce that $\sigma(\mathcal{A})$ is homeomorphic to $\mathcal{U}$ and that one can identify the Gelfand transform with a discrete Fourier transform.
5. Let $\mathcal{W}$ be the set of continuous 2π-periodic functions whose Fourier series converges absolutely. Prove that if f has no zeros, then $1/f \in \mathcal{W}$.

Exercise 2 – ***Commutant and multiplication operators.*** Let $\mathcal{A}_X = \{\, M_f \in \mathcal{B}(L^2(X)) : f \in L^\infty(X) \,\}$ where M_f denotes the multiplication operator by $f(x)$.

1. If $X \subset \mathbb{R}^d$ be compact, Prove that $\mathcal{A}_X$ is its own commutant in $\mathcal{B}(L^2(X))$.
2. Prove the same result if $X = \mathbb{R}^d$.
3. Using the previous question, find the commutant of the set $\mathcal{T}_2 = \{\, \tau_x \in \mathcal{B}(L^2(\mathbb{R}^d)) : x \in \mathbb{R}^d \,\}$, where $\tau_x\psi(y) = \psi(x - y)$ and of the set $\mathcal{T}_2 \cup \mathcal{A}_{\mathbb{R}^d}$
4. Find the commutant of $\mathcal{T}_1 = \{\, \tau_x \in \mathcal{B}(L^1(\mathbb{R}^d)) : x \in \mathbb{R}^d \,\}$.

Exercise 3 – ***Commutant and shift.***

1. Find the commutant and the bicommutant of the set $\{\, \mathrm{Id}, \tau_l, \tau_r \,\}$ seen as a subset of $\mathcal{B}(\ell^2(\mathbb{N}))$, where τ_l and τ_r are the left and right shift operators.
2. Find the commutant of the bilateral shift $\tau \in \mathcal{B}(\ell^2(\mathbb{Z}))$ defined by $(\tau u)_n = u_{n+1}$.

Exercise 4 – ***Closed ideals of*** $C^0(X)$. Let $\mathcal{A} = C^0(X)$ for some compact Hausdorff space X.

1. Prove that for any proper ideal $\mathcal{I} \subset \mathcal{A}$ there exists $x \in X$ such that $\forall f \in \mathcal{I}, f(x) = 0$.
2. Prove that $\mathcal{I}$ is maximal iff the above property is true for exactly one $x \in X$.
3. Show that there is a bijection between the set of compact subsets of X and the set of closed ideals of $C^0(X)$.

TD 12 - Unbounded operators

Let H be a Hilbert space.

Exercise 1 – ***Commutant and multiplication operators.*** Let $\mathcal{A}_X = \{ M_f \in \mathcal{B}(L^2(X)) : f \in L^\infty(X) \}$ where M_f denotes the multiplication operator by $f(x)$.

1. If $X \subset \mathbb{R}^d$ be compact, Prove that $\mathcal{A}_X$ is its own commutant in $\mathcal{B}(L^2(X))$.
2. Prove the same result if $X = \mathbb{R}^d$.
3. Using the previous question, find the commutant of the set $\mathcal{T}_2 = \{ \tau_x \in \mathcal{B}(L^2(\mathbb{R}^d)) : x \in \mathbb{R}^d \}$, where $\tau_x \psi(y) = \psi(x-y)$ and of the set $\mathcal{T}_2 \cup \mathcal{A}_{\mathbb{R}^d}$
4. Find the commutant of $\mathcal{T}_1 = \{ \tau_x \in \mathcal{B}(L^1(\mathbb{R}^d)) : x \in \mathbb{R}^d \}$.

Exercise 2 – ***Unbounded operators and domains.***

1. Let A be the operator given by $A\psi = (1-\Delta)\,\psi$ with domain $D(A) = C_c^\infty(\mathbb{R}^d)$. Prove that one cannot extend it as a bounded operator on $L^2(\mathbb{R}^d)$. What is the closure of A?
2. Consider the Banach space $X = C^0([0,1])$ endowed with the L^∞ norm and the operators A_{max}, A_{min}, A_k, and A_{00} all defined by the same action
$$A\varphi = \frac{\mathrm{d}\varphi}{dx}$$
but with different domains $D(A_{\text{max}}) = C^1([0,1])$, $D(A_k) = \{ \varphi \in C^1([0,1]) : \varphi(0) = k\varphi(1) \}$, $D(A_{\text{min}}) = C_c^\infty(]0,1[)$, and $D(A_{00}) = \{ \varphi \in C^1([0,1]) : \varphi(0) = \varphi(1) = 0 \}$. Study the injectivity, surjectivity and closure of these operators.

Exercise 3 – ***A Non-closable operator.*** Let e_n be an orthonormal basis of H, D be the set of finite linear combinations of the e_n and $e_0 \in H \setminus D$. Let A be an unbounded operator on H with domain D defined by

$$A(e_0) = e_0 \quad \text{and} \quad \forall k \in \mathbb{N}^*, A(e_k) = 0\,.$$

Prove that the closure of the graph of A is not the graph of a linear operator, and so that A is not closable.

Exercise 4 – ***Some properties of unbounded operators.*** Let A be an injective unbounded operator on H with domain $D(A)$. Consider the following statements about S.

(a) A is closed.

(b) $\operatorname{Ran}(A)$ is dense.

(c) $\operatorname{Ran}(A)$ is closed.

(d) For some constant $C > 0$, $\forall \psi \in D(A)$, $\|A\psi\| \geq C\,\|\psi\|$.

Prove that (a,b,c) implies (d), (b,c,d) implies (1), and (a) and (d) imply (c). Prove that if A verifies (a,b) and $B \in \mathcal{B}(H)$, then $A+B$ is closed.

Exercise 5 – ***Commutators.***

1. Prove that there is no operators $A, B \in \mathcal{B}(H)$ such that $[A,B] = \mathrm{Id}$.
2. Is it still true if A and B are allowed to be unbounded?