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Dynamique de systèmes à grand nombre de particules et systèmes dynamiques

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Introduction

1 Motivations et cadre de la thèse

Cette thèse porte sur l'étude de la dynamique de systèmes physiques hors équilibre contenant un grand nombre de particules. L'objectif est d'étudier mathématiquement et de façon quantitative le comportement de modèles physiques décrivant un ensemble de corps en interaction et soumis à des forces externes et internes. De tels modèles conduisent à des équations aux dérivées partielles qui permettent à la fois d'étudier le comportement des grandeurs mesurables du phénomène, mais aussi de créer par la suite des algorithmes pour la modélisation par ordinateur. Les équations étudiées dans cette thèse se retrouvent dans plusieurs branches de la physique et aussi à des échelles très différentes. On peut citer par exemple la physique statistique et l'astrophysique, avec en particulier l'étude des gaz, des plasmas et des galaxies, la physique atomique, moléculaire et des semi-conducteurs et la mécanique des fluides. Mais les applications vont aujourd'hui au-delà de la physique puisqu'on trouve aussi des applications en biologie, en économie et en finance.

Historiquement, ces modèles se sont aussi développés à différentes époques correspondant à l'avancée de la compréhension scientifique. Ainsi, dès la fin du XVII^{e} siècle, I. Newton, dans ses *Philosophiae naturalis principia mathematica* [175] énonce des lois qui permettent déjà de formuler mathématiquement les équations du mouvement de N corps en interaction et soumis à des forces externes. Elles apportent un début de modélisation des fluides à un **niveau microscopique**.

Par ailleurs, au XVIII^{e} et au XIX^{e} siècle, des équations décrivant des milieux continus au **niveau macroscopique** voient le jour, telles que les équations hydrodynamiques d'Euler et de Navier-Stokes et l'équation de la chaleur. L'arrivée de l'équation cinétique de Boltzmann en 1872 dans le paysage scientifique fait apparaître un nouveau **niveau mésoscopique** intermédiaire, qui décrit le comportement de la densité des particules dans l'espaces des phases. C'est dans ce contexte que D. Hilbert énonce en 1900 dans son 6^{e} problème le besoin d'axiomatiser mathématiquement la physique et en particulier le lien entre les modèles discrets et continus : *« le livre de M. Boltzmann sur les Principes de la Mécanique nous incite à établir et à discuter du point de vue mathématique d'une manière complète et rigoureuse les méthodes basées sur l'idée de passage à la limite, et qui de la conception atomique nous conduisent aux lois du mouvement des continua. »* (D. Hilbert, [119]).

Cependant, le XX^{e} siècle s'accompagne de l'apparition de nouvelles théories physiques dont la mécanique quantique qui vient préciser le comportement aux très petites échelles et introduit donc un autre **niveau de description quantique**, ainsi que la relativité générale qui traite des grandes échelles mais qui n'est pas traitée dans cette thèse. Il est donc important de comprendre en quel sens toutes ces descriptions sont similaires et ce en quoi elles diffèrent.

Aujourd'hui, bien que l'on pourrait se dire que la mécanique quantique, voire une théorie plus avancée, est le modèle le plus juste de par sa précision aux petites échelles, il convient aussi d'adapter la complexité des modèles pour qu'ils soient simples à étudier et rapides à simuler. Un exemple de limitation des modèles trop détaillés est ce qu'on appelle la malédiction de la dimension, qui est le fait qu'un système pour lequel chaque particule est prise en compte conduit mathématiquement à une explosion exponentielle du nombre de degrés de liberté, et donc du temps de calcul, ce qui empêche dans la pratique de modéliser un système par un modèle qui n'est pas à la bonne échelle. Chacun des modèles a donc une utilité, mais il faut pour cela bien comprendre quelle est sa place.

Il est intéressant de noter que les progrès de la physique dans la compréhension des descriptions du monde à toutes ces échelles sont aujourd'hui utilisés pour définir les unités du système international [1]. Ainsi à partir de mai 2019, la **constante de Planck**

$$h = 6{,}626\,070\,15 \times 10^{-34}\,\mathrm{kg\,m^2\,s^{-1}}$$

associée à l'échelle de description quantique sert de définition pour le kilogramme pour mesurer une masse, la **constante de Boltzmann**

$$k_B := 1{,}380\,649 \times 10^{-23}\,\mathrm{kg\,m^2\,s^{-2}\,K^{-1}}$$

associée à l'échelle mésoscopique sert de définition pour le Kelvin pour mesurer une température, la **constante d'Avogadro**

$$N_A = 6{,}022\,140\,76 \times 10^{23}\,\mathrm{mol^{-1}}$$

associée à l'échelle macroscopique sert de définition à la mole qui mesure la quantité de matière, et la **vitesse de la lumière** dans le vide

$$c = 299\,792\,458\,\mathrm{m\,s^{-1}}$$

associée à l'échelle relativiste sert de définition au mètre.

2 Présentation et dérivation des modèles

Tous les travaux présentés ici étudient des équations aux dérivées partielles qui décriront l'évolution d'une densité de particules dans l'espace. Ces particules, selon le modèle, peuvent aussi représenter autre chose, comme des bactéries par exemple dans le Chapitre 6. Dans tous les cas on désignera par $x \in \mathbb{R}^d$ la variable d'espace pour un entier $d \geq 1$, par $t \in \mathbb{R}_+$ la variable temporelle et par $v \in \mathbb{R}^d$ la variable de vitesse. On présente dans cette section tout un ensemble de modèles qui décrivent des particules en interaction et soumises à des forces externes et internes pour que le lecteur puisse comprendre où se situent les modèles qui sont étudiés dans la suite et comment ils sont dérivés.

2.1 Modèle à N particules ponctuelles

Le premier type de modèle que l'on peut avoir envie d'envisager pour étudier le comportement d'un système physique composé d'un grand nombre de particules en interaction est le modèle de N points définis par leur position $X_\mathrm{i} = X_\mathrm{i}(t) \in \mathbb{R}^d$ et leur impulsion $P_\mathrm{i} = P_\mathrm{i}(t)$ pour $\mathrm{i} \in [\![1, N]\!]$. Ce modèle ne sera pas étudié dans le reste de la thèse mais permet de comprendre le cadre et certaines propriétés des autres modèles. Si les points sont soumis à des forces E_i, d'après les lois de Newton, un tel système vérifie alors

$$\begin{aligned} \mathrm{d}X_\mathrm{i} &= \frac{1}{m} P_\mathrm{i} \,\mathrm{d}t \\ \mathrm{d}P_\mathrm{i} &= E_\mathrm{i}(X) \,\mathrm{d}t, \end{aligned}$$

où m représente la masse des particules et $X = (X_\mathrm{i})_{\mathrm{i} \in [\![1,N]\!]}$. Dans la suite on supposera que l'unité est choisie de telle manière à ce que la masse des particules vaille $m = 1$ et l'on identifiera donc la vitesse et l'impulsion. Dans le cas où ces forces découlent d'un potentiel, c'est-à-dire que $E_\mathrm{i} = -\nabla V_\mathrm{i}$, ce système peut se réécrire au moyen du Hamiltonien

$$H_N(X, P) := \frac{1}{2} \sum_{\mathrm{i}=1}^{N} |P_\mathrm{i}|^2 + \sum_{\mathrm{i}=1}^{N} V_\mathrm{i}(X),$$

puisqu'il vérifie alors les équations de Hamilton

$$\begin{aligned} \mathrm{d}X_\mathrm{i} &= \nabla_{P_\mathrm{i}} H_N(X_\mathrm{i}, P_\mathrm{i}) \,\mathrm{d}t \\ \mathrm{d}P_\mathrm{i} &= -\nabla_{X_\mathrm{i}} H_N(X_\mathrm{i}, P_\mathrm{i}) \,\mathrm{d}t. \end{aligned}$$

On peut alors montrer que la solution conserve les volumes dans l'espace des phases $\mathbb{R}^{2dN}$ ainsi que le Hamiltonien. Dans le cadre d'un système de particules en interaction au travers d'un potentiel d'interaction $K(x - y)$ et soumis à un potentiel externe $U(x)$, on a en particulier

$$V_\mathrm{i}(X) = U(X_\mathrm{i}) + \sum_{\mathrm{j}=1}^{N} K(X_\mathrm{i} - X_\mathrm{j}).$$

Ce système peut en fait se réécrire sous forme d'une équation aux dérivées partielles en considérant la mesure empirique associée aux particules dans l'espace des phases

$$f_N = \sum_{\mathrm{i}=1}^{N} \delta_{(X_\mathrm{i}, P_\mathrm{i})},$$

où $\delta_{(x,v)}$ désigne la mesure de Dirac au point $(x, v) \in \mathbb{R}^{2d}$. En effet, si K est assez régulier, cette mesure empirique vérifie alors au sens des distributions **l'équation de Liouville**

$$\frac{\partial f_N}{\partial t} + v \cdot \nabla_x f_N + E(x) \cdot \nabla_v f_N = 0. \tag{1}$$

2.2 Équation de Vlasov

Cette équation décrit au niveau mésoscopique le cas où les particules interagissent entre elles via un potentiel d'interaction mais que les collisions sont négligeables. Elle intervient notamment dans la description des plasmas, qui apparaissent par exemple en astrophysique pour décrire les étoiles, les gaz interstellaires et le vent solaire, pour certains phénomènes atmosphériques comme les éclairs et les aurores boréales, et aussi dans de nombreuses applications industrielles. Le plasma est un état de la matière dans lequel les particules sont dans un état assez dilué, comme pour un gaz, mais dans lequel il y a aussi un nombre non négligeable de particules chargées. Dans ce cas, le potentiel d'interaction créé par une charge $q \in \mathbb{R}$ située en $0 \in \mathbb{R}^d$ s'obtient par la **loi de Coulomb** et s'écrit

$$K(x) = \frac{q}{4\pi\varepsilon_0}\frac{1}{|x|},$$

où ε_0 est la permittivité du vide.

Un autre cas d'application est celui d'un système de corps en interaction gravitationnelle, auquel cas le potentiel créé par une masse $m \in \mathbb{R}_+$ s'écrit

$$K(x) = -\frac{G\,m}{|x|},$$

où G est la constante universelle de gravitation.

Notons que le potentiel gravitationnel est toujours attractif alors que le potentiel Coulombien peut être aussi bien attractif que répulsif en fonction du signe des charges considérées. En particulier, dans le cas où les deux signes de charges sont présents, un modèle souvent utilisé en physique des plasmas est le **potentiel de Yukawa**

$$K(x) = \frac{q}{4\pi\varepsilon_0}\frac{e^{-|x|/\lambda_D}}{|x|}, \tag{Yukawa}$$

où λ_D désigne la longueur caractéristique de l'effet d'écran, appelée longueur de Debye. Bien sûr, d'autres type de potentiels peuvent aussi apparaître selon le problème étudié.

Du fait du grand nombre de particules et de l'échelle temporelle considérée, on peut par exemple effectuer un changement de variable $t \leftrightarrow \frac{t}{\sqrt{N}}$ et $v \leftrightarrow v\sqrt{N}$ de telle sorte que l'équation de Liouville (1) devienne

$$\frac{\partial f_N}{\partial t} + v \cdot \nabla_x f_N + \frac{1}{N}\,E(x) \cdot \nabla_v f_N = 0.$$

Dans cette échelle, on a finalement juste modifié l'interaction d'un facteur N^{-1}. On a donc une force effective $E_N(x) = \frac{1}{N}E(x)$ et le potentiel d'interaction associé s'écrit alors

$$V_N(x) = \frac{1}{N}\sum_{\mathrm{j}=1}^{N} K(x - X_{\mathrm{j}}).$$

Le cadre de la limite de champ moyen est alors celui tel que dans la limite $N \to \infty$, la densité spatiale, c'est-à-dire la répartition des X_{i}, converge vers une mesure absolument continue qui s'écrit

$$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \tag{2}$$

et le potentiel converge vers un potentiel moyen dépendant uniquement de cette densité, qui s'écrit

$$V_N(t,x) \underset{N\to\infty}{\longrightarrow} V(t,x) = \int_{\mathbb{R}^d} K(x-y)\rho(y)\,\mathrm{d}y.$$

On obtient alors formellement l'**équation de Vlasov** (non-linéaire) qui s'écrit

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f - (\nabla K * \rho)\cdot\nabla_v f = 0, \tag{Vlasov}$$

où $*$ désigne le produit de convolution sur $\mathbb{R}^d$.

2.3 Modèles quantiques

En mécanique quantique, la notion de particule ponctuelle est remplacée par celle de **fonction d'onde** qui est une fonction $\psi \in L^2(\mathbb{R}^d,\mathbb{C})$, l'espace des fonctions de carré intégrable de $\mathbb{R}^d$ dans $\mathbb{C}$, vérifiant

$$\int_{\mathbb{R}^d} |\psi(x)|^2\,\mathrm{d}x = 1.$$

Le carré du module $\rho(x) = |\psi(x)|^2$ peut être pensé comme représentant la distribution de probabilité de trouver la particule au point x, alors que le carré du module de la transformée de Fourier de ψ, noté $|\widehat{\psi}(\xi)|^2$, représente la probabilité que la particule ait la vitesse ξ. En présence d'un potentiel V, la fonction d'onde vérifie l'**équation de Schrödinger**

$$i\hbar\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\,m}\,\Delta_x\psi + V(t,x)\,\psi, \tag{3}$$

où $\psi = \psi(t,x)$ et $\hbar = \frac{h}{2\pi}$ est la constante de Planck réduite. Pour N particules, la fonction d'onde est définie sur l'espace produit $\mathbb{R}^{dN}$. En définissant le Hamiltonien quantique comme l'opérateur

$$\mathsf{H}_N = \frac{1}{2}\sum_{\mathrm{i}=1}^N |\mathsf{p}_{\mathrm{i}}|^2 + \sum_{\mathrm{i}=1}^N V(x_{\mathrm{i}}),$$

où $\mathsf{p}_{\mathrm{i}} = -i\hbar\nabla_{x_{\mathrm{i}}}$, $|\mathsf{p}_{\mathrm{i}}|^2 = -\hbar^2\Delta_{x_{\mathrm{i}}}$ et $V(x_{\mathrm{i}})$ désigne l'opérateur de multiplication par V évalué en la i^{e} variable d'espace, on obtient alors l'**équation de Schrödinger à N corps**, ou équation de Liouville quantique

$$i\hbar\,\frac{\partial\Psi_N}{\partial t} = \mathsf{H}_N\Psi_N. \tag{4}$$

La dénomination d'équation de Liouville quantique se comprend d'autant mieux lorsqu'on regarde ce qu'il se passe non plus pour une fonction d'onde mais pour l'opérateur de projection sur celle-ci, appelé opérateur densité et noté $\boldsymbol{\rho}_N := |\Psi_N\rangle\langle\Psi_N|$. Les notations "*bra*" $\langle\cdot|$ et "*ket*" $|\cdot\rangle$ désignent respectivement la première et la deuxième "*moitié*" du produit scalaire sur $L^2(\mathbb{R}^{dN},\mathbb{C})$

$$\langle\Psi|\varphi := \langle\Psi,\varphi\rangle_{L^2(\mathbb{R}^{dN},\mathbb{C})} = \int_{\mathbb{R}^{dN}} \overline{\Psi(x_1,\dots,x_N)}\,\varphi(x_1,\dots,x_N)\,\mathrm{d}x_1\dots\mathrm{d}x_N$$
$$|\Psi\rangle := \Psi(x_1,\dots,x_N).$$

On obtient alors l'**équation de Liouville-Von Neumann** qui s'écrit

$$i\hbar\frac{\mathrm{d}\boldsymbol{\rho}_N}{\mathrm{d}t} = [\mathsf{H}_N,\boldsymbol{\rho}_N]\,,$$

où $[\cdot,\cdot]$ désigne le commutateur défini par $[\mathsf{A},\mathsf{B}] = \mathsf{AB}-\mathsf{BA}$. Cette équation est à mettre en parallèle avec l'équation de Liouville classique (1) qui peut se réécrire

$$\frac{\partial f_N}{\partial t} = \{H_N, f_N\},$$

où la notation $\{\cdot,\cdot\}$ désigne les crochets de Poisson définis par $\{f,g\} := \nabla_x f\cdot\nabla_v g - \nabla_v f\cdot\nabla_x g$. Ces opérateurs densité qui sont des matrices de rang 1 sont appelés **états purs**, et plus généralement, on peut considérer des états appelés **mélanges statistiques** qui s'obtiennent comme une combinaison linéaire d'états purs sous la forme

$$\boldsymbol{\rho}_N := \sum_{\mathrm{j}\in\mathbb{N}} \lambda_{\mathrm{j}}\,|\Psi_{N,\mathrm{j}}\rangle\langle\Psi_{N,\mathrm{j}}|,$$

où les λ_{j} forment une suite de réels positifs. Ces états ne peuvent pas toujours se mettre sous la forme d'états purs. En particulier, si les $\Psi_{N,\mathrm{j}}$ forment une famille orthonormée et que la somme des λ_{j} vaut 1, alors on obtient $\boldsymbol{\rho}_N^2 = \boldsymbol{\rho}_N$ si et seulement si tous les λ_{j} sauf un sont nuls. Dans ce cas plus général la densité spatiale qui généralise $\rho(x) = |\psi(x)|^2$ est donnée par

$$\rho_N(x) := \sum_{\mathrm{j}\in\mathbb{N}} \lambda_{\mathrm{j}}\,|\Psi_{N,\mathrm{j}}(x)|^2\,.$$

De même que dans le cas classique des modèles à N particules, lorsque le potentiel est un potentiel d'interaction, on peut considérer la limite de champ moyen. On retrouve alors une équation à une seule variable d'espace, l'**équation de Hartree**, qui s'écrit

$$i\hbar\frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t} = [\mathsf{H},\boldsymbol{\rho}]\,, \tag{Hartree}$$

où $\mathsf{H} = \frac{|\mathrm{p}|^2}{2} + V$ avec $V = K * \rho$, l'opérateur densité et la densité spatiale étant définis respectivement par

$$\boldsymbol{\rho} = \sum_{\mathrm{j}\in\mathbb{N}} \lambda_{\mathrm{j}}\,|\psi_{\mathrm{j}}\rangle\langle\psi_{\mathrm{j}}|$$
$$\rho(x) = \sum_{\mathrm{j}\in\mathbb{N}} \lambda_{\mathrm{j}}\,|\psi_{\mathrm{j}}(x)|^2\,. \tag{5}$$

2.4 Équation cinétiques collisionnelles

Dans le cas où les potentiels d'interaction deviennent singuliers, c'est-à-dire très grands au voisinage des particules et que les particules peuvent se retrouver suffisamment proches les unes des autres, les modèles de champ moyen ne sont plus un bon choix puisque l'interaction entre paires de particules proches devient dominante par rapport aux interactions avec l'ensemble des autres particules. On appelle ce type d'interaction des collisions. Le cas le plus simple du point de vue conceptuel correspond à celui des boules de billard, où le potentiel peut être modélisé comme étant infini au niveau des boules et nul à l'extérieur. Ce modèle est généralement appelé modèle des sphères dures. Il y a alors un saut brusque dans le comportement des vitesses à chaque collision. On obtient des équations cinétiques qui s'écrivent

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f),$$

où $Q(f,f)$ est un terme de collision quadratique, telles que l'**équation de Boltzmann** [43] et l'**équation de Landau** [137]. Dans le cas de l'équation de Boltzmann, le terme de collision prend la forme

$$Q_B(f,f)(x,v) = \iint_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{B}(|v_* - v|, \sigma)\,(f' f'_* - f f_*)\,\mathrm{d}v_*\,\mathrm{d}\sigma, \tag{6}$$

où l'on a noté $f' = f(x,v')$, $f_* = f(x,v_*)$ et $f'_* = f(x,v'_*)$. Ici v et v_* sont les vitesses de deux particules avant collision et v' et v'_* sont les vitesses des particules après collision et le vecteur unitaire $\sigma = \frac{v'-v'_*}{|v'-v'_*|} \in \mathbb{S}^{d-1}$ donne les différent angles possibles après collision. B est le noyau de collision. Il est à noter que ce noyau de collision devient singulier lorsque les forces deviennent à longue portée.

Dans le cas où l'on considère un ensemble de particules diluées dans un milieu à l'équilibre thermodynamique dont on connait le profil des vitesses $F(v)$, l'**équation de Boltzmann linéaire** donne un modèle simplifié qui s'écrit avec des notations similaires

$$\frac{\partial f}{\partial t} + \mathsf{T} f = \mathsf{L} f := \int_{\mathbb{R}^d} \mathrm{b}(x,v,v_*)\,(f_* F - f F_*)\,\mathrm{d}v_*, \tag{7}$$

où $\mathsf{T} = v \cdot \nabla_x$ désigne l'opérateur de transport, qui peut aussi être remplacé par exemple par $\mathsf{T} = v \cdot \nabla_x + E(x) \cdot \nabla_v$ si on veut prendre en compte l'effet d'une force extérieure ou de champ moyen.

Une autre approximation très utilisée en physique des plasmas est l'**équation de Fokker-Planck cinétique**, c'est-à-dire le cas où

$$\mathsf{L} f = \mathrm{div}_v\left(F\,\nabla_v\left(\frac{f}{F}\right)\right) = \Delta_v f + \mathrm{div}_v(E_v(v) f). \tag{8}$$

Remarquons que l'on peut écrire cette équation d'un point de vue Lagrangien semblable à celui du modèle à N particules en utilisant la théorie des probabilités. Dans ce cas, $X(t)$ et $P(t)$ deviennent des variables aléatoires et on obtient une équation stochastique qui s'écrit

$$\begin{aligned} \mathrm{d}X &= P\,\mathrm{d}t \\ \mathrm{d}P &= -E_v(P)\,\mathrm{d}t + \mathrm{d}B, \end{aligned}$$

où $B = B(t)$ désigne le mouvement brownien. On voit que le Laplacien, qui correspond au mouvement Brownien, peut être vu comme étant un terme aléatoire dû à l'agitation thermique alors que la force E_v peut être vue comme correspondant à une force de friction sur les vitesses. On peut retrouver des modèles de type Fokker-Planck en effectuant une limite dite *de collisions rasantes* à partir d'une équation de Boltzmann Linéaire (voir par exemple [156]).

2.5 Le Laplacien fractionnaire

Un détail important des opérateurs de collision apparaissant de type intégraux tels (6) ou (7) est de savoir si le noyau de collision B ou b devient singulier en certains points. Dans le cas de l'opérateur de Boltzmann (6), on obtient en effet de telles singularités dès que l'interaction microscopique à l'origine de la collision est à longue portée. Bien que ces singularités sont à l'origine de certaines difficultés techniques dans la définition et la manipulation des opérateurs, elles peuvent cependant aussi être responsables de gains de régularité pour les solutions des équations associées. Un modèle simple pour comprendre ce type de régularisation est celui du **Laplacien fractionnaire** défini pour $\alpha \in]0,2[$ par

$$\Delta^{\frac{\alpha}{2}} u(x) := c_{d,\alpha}\ \mathrm{vp} \int_{\mathbb{R}^d} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}}\,\mathrm{d}y = c_{d,\alpha} \lim_{\varepsilon\to 0^+} \int_{|y-x|>\varepsilon} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}}\,\mathrm{d}y. \tag{9}$$

pour une certaine constante $c_{d,\alpha} > 0$ dépendant de d et α. Remarquons qu'il peut être écrit sous la forme d'un opérateur de Boltzmann Linéaire L comme défini par l'équation (7) puisque l'on a pour ce dernier, en prenant $u := fF^{-1}$,

$$\mathsf{L}\,(uF) = \int_{\mathbb{R}^d} \mathrm{b}(x,v,v_*) F(v_*) F(v)\,(u(v_*) - u(v))\,\mathrm{d}v_*,$$

ce qui donne le Laplacien fractionnaire dès que $\mathrm{b}(x,v,v_*) = F(v)F(v_*)|v_* - v|^{-(d+\alpha)}$.

Cet opérateur n'est cependant pas qu'un exemple jouet d'opérateur faisant apparaître une singularité dans un opérateur intégral, et il se retrouve à l'interaction entre de nombreux domaines et théories mathématiques. Tout d'abord, comme sa notation le laisse deviner, il est une puissance fractionnaire du laplacien classique puisqu'il vérifie

$$\Delta^{\frac{\alpha}{2}} = -(-\Delta)^{\frac{\alpha}{2}},$$

où $(-\Delta)^{\frac{\alpha}{2}}$ désigne la puissance de l'opérateur positif $-\Delta$ sur $\mathbb{R}^d$ au sens de la théorie spectrale des opérateurs autoadjoints. En particulier, il peut aussi s'écrire sous la forme d'un multiplicateur de Fourier au même titre que le Laplacien classique, ce qui donne

$$\mathcal{F}(\Delta^{\frac{\alpha}{2}} u)(y) = -|2\pi y|^{\alpha}\,\mathcal{F}(u)(y),$$

où l'on a pris la définition suivante de la transformée de Fourier

$$\mathcal{F}(u)(y) = \int_{\mathbb{R}^d} e^{-2i\pi x\cdot y} u(x)\,\mathrm{d}x.$$

Notons que la formulation intégrale découle assez aisément de la formulation sous forme de multiplicateur de Fourier puisque par une simple considération de changement d'échelle[1] (voir aussi par exemple [200, 148]) on peut deviner la formule suivante pour tout $a \in]0,d[$

$$\mathcal{F}\left(\frac{1}{\omega_a |x|^a}\right) = \frac{1}{\omega_{d-a}|x|^{d-a}},$$

où $\omega_a = \frac{2\pi^{\frac{a}{2}}}{\Gamma\left(\frac{a}{2}\right)}$, Γ désignant la fonction Gamma. En particulier, $\omega_d = |\mathbb{S}^{d-1}|$ n'est rien d'autre que le volume de la sphère unité de dimension $d-1$. En prenant la dérivée au sens des distributions, on obtient alors aussi la formule pour $\alpha = -a \in (-2,0)$ où le membre de droite doit être compris au sens des valeurs principales, c'est-à-dire comme la distribution définie par

$$\left\langle \mathrm{vp}\left(\frac{1}{|x|^{d+\alpha}}\right), \varphi \right\rangle_{\mathcal{D}',\mathcal{D}} = \mathrm{vp}\int_{\mathbb{R}^d} \frac{\varphi(x)-\varphi(0)}{|x|^{d+\alpha}}\,\mathrm{d}x.$$

On obtient alors la formule intégrale du Laplacien fractionnaire en utilisant les propriétés de la transformée de Fourier en écrivant

$$\begin{aligned}\Delta^{\frac{\alpha}{2}} u(x) &= \mathcal{F}^{-1}\left(-|2\pi y|^{\alpha}\,\mathcal{F}(u)\right) = -(2\pi)^{\alpha}\,\mathcal{F}^{-1}\left(|y|^{\alpha}\right) * u \\ &= \mathrm{vp}\left(\frac{c_{d,\alpha}}{|x|^{d+\alpha}}\right) * u(x),\end{aligned}$$

où $*$ désigne le produit de convolution sur $\mathbb{R}^d$ et où on comprend maintenant que $c_{d,\alpha} = -\frac{(2\pi)^{\alpha}\omega_{-\alpha}}{\omega_{d+\alpha}} > 0$, ce qui donne exactement la formule (9). Cela permet aussi de remarquer que le Laplacien fractionnaire n'est rien d'autre qu'un produit de convolution par une fonction de type puissance. Cependant, cette fonction puissance n'étant plus intégrable, elle devient une distribution d'ordre strictement positif, ce qui implique qu'elle n'a donc plus de signe[2], ce qui est cohérent avec son lien avec les opérateurs différentiels.

Le Laplacien fractionnaire est aussi un cas particulier des opérateurs de Lévy qui sont les générateurs infinitésimaux des **processus de Lévy**. Ces derniers sont des processus stochastiques parmi lesquels on trouve par exemple le mouvement Brownien et le processus de Poisson, et qui peuvent être discontinus en temps. Les processus de Lévy associés au Laplacien fractionnaire sont des processus α-stables, ce qui signifie que si X_t est un processus de Lévy, alors $X_{c^{\alpha}t}$ a la même loi que cX_t. Ils peuvent en cela être vus comme des généralisations du mouvement Brownien qui est un processus 2-stable. On trouve de nombreuses applications de ces processus à la finance.

[1] Ceci ne donne bien sûr pas la constante, qui peut s'obtenir en multipliant par une gaussienne, en intégrant puis en calculant explicitement les formules obtenues.

[2] Du moins au voisinage de 0.

Une généralisation naturelle de l'équation de Fokker-Planck cinétique définie par (8) conservant l'interprétation probabiliste des vitesses soumises à une dérive et à un processus de saut $\mathfrak{a}$-stable est alors l'**équation de Fokker-Planck fractionnaire cinétique** suivante

$$\frac{\partial f}{\partial t} + \mathsf{T}f = \mathsf{L}f = \Delta_v^{\frac{\mathfrak{a}}{2}} f + \mathrm{div}_v(E_v(v) f), \tag{10}$$

pour un certain $\mathfrak{a} \in]0, 2[$.

2.6 Limites de diffusion

Les équations au niveau macroscopique n'étudient plus les distributions des particules dans l'espace des phases mais directement les grandeurs mesurables en chaque position $x \in \mathbb{R}^d$, telles que la densité spatiale $\rho(x)$ définie par l'équation (2) dans le cas cinétique et l'équation (5) dans le cas quantique. On peut aussi définir la vitesse macroscopique et la température respectivement par

$$\begin{aligned}\rho\, u &:= \int_{\mathbb{R}^d} v\, f(\cdot, v)\, \mathrm{d}v \\ \rho\, T &:= \frac{m}{3\, k_B} \int_{\mathbb{R}^d} |v - u|^2\, f(\cdot, v)\, \mathrm{d}v.\end{aligned}$$

Les différentes équations présentées vérifient alors toutes la conservation locale de la densité spatiale qui s'écrit

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho u) = 0. \tag{11}$$

On peut alors chercher à trouver d'autres équations sur ρ et u pour obtenir un système fermé d'équations. Ce type de stratégie permet de dériver les modèles hydrodynamiques, comme les équations d'Euler et de Navier-Stokes, à partir de l'équation de Boltzmann en considérant différents changements d'échelle.

Pour une équation cinétique avec un opérateur de collision linéaire L, on considère ici le changement d'échelle diffusif qui s'écrit

$$\varepsilon^{\alpha} \frac{\partial f}{\partial t} + \varepsilon v \cdot \nabla_v f = \mathsf{L}f, \tag{12}$$

où $\alpha \in]0, 2]$ dépend de l'état d'équilibre et de l'opérateur L en général, avec $\alpha = 2$ si l'état d'équilibre F de l'opérateur L a suffisamment de moments bornés. Il a été montré que ce type d'équation converge alors vers l'équation de la chaleur ou sa version factionnaire. Plus précisément, la solution $f_\varepsilon(t, x, v)$ de l'équation (12) converge lorsque $\varepsilon \to 0$ vers $\rho(t, x)\, F(v)$ où ρ est solution de

$$\frac{\partial \rho}{\partial t} = \kappa\, \Delta^{\frac{\alpha}{2}} \rho,$$

pour un certain $\kappa > 0$. Voir P. Degond, T. Goudon et F. Poupaud [73] pour le cas de la limite diffusion classique $\alpha = 2$ en prenant pour L un opérateur de Boltzmann Linéaire, A. Mellet, C. Mouhot et S. Mischler [161, 160] pour le cas de la limite de diffusion fractionnaire avec le même opérateur de collision, et G. Lebeau et M. Puel [141] et N. Fournier et C. Tardif [94, 93] pour le cas de l'opérateur de collision de Fokker-Planck. De même si on ajoute un champ de force dans la variable d'espace on peut prendre le changement d'échelle suivant

$$\varepsilon^\alpha \frac{\partial f}{\partial t} + \varepsilon v \cdot \nabla_v f + \varepsilon^{\alpha-1} E(x) \cdot \nabla_v f = \mathsf{L} f.$$

Si on prend $\mathsf{L}f = \Delta_v^{\frac{\alpha}{2}} f + \mathrm{div}_v(vf)$, alors comme démontré par P. Aceves-Sanchez et L. Cesbron [2], on obtient à la limite que ρ vérifie

$$\frac{\partial \rho}{\partial t} = \Delta^{\frac{\alpha}{2}} \rho + \mathrm{div}(E\rho),$$

du moins sous certaines conditions sur E. Il est intéressant de regarder la façon dont sont reliés l'opérateur L, l'état d'équilibre et la puissance du Laplacien fractionnaire qui apparaît à la limite. Dans le cas où on peut écrire l'opérateur de Boltzmann Linéaire (7) sous la forme

$$\mathsf{L} f = \int_{\mathbb{R}^d} \mathrm{b}(v, v_*) \left(f_* F - f F_* \right) \mathrm{d}v_* = \mathcal{B}(f) - \nu(v) f, \tag{13}$$

alors si on définit $\langle v \rangle := \sqrt{1 + |v|^2}$ et on prend

$$\begin{aligned} F(v) &= \frac{C_\gamma}{\langle v \rangle^{d+\gamma}} \\ \nu(v) &\underset{v\to\infty}{\sim} |v|^\beta, \end{aligned} \tag{14}$$

avec $\gamma > 0$ et $\beta < \gamma$, on obtient que α est donné par[3]

$$\begin{aligned} \alpha &= \frac{\gamma - \beta}{1 - \beta} && \text{si } \gamma + \beta < 2 \\ \alpha &= 2 && \text{si } \gamma + \beta \geq 2. \end{aligned}$$

Pour le cas de l'opérateur de Fokker-Planck (8), le même résultat s'obtient en prenant $\beta = -2$ et pour le cas de l'opérateur de Fokker-Planck fractionnaire (10), on s'attend au même résultat en prenant $\beta = \gamma - \mathfrak{a}$. Ceci est résumé en Figure 1.

Le fait d'avoir $\beta = -2$ pour l'opérateur de Fokker-Planck (8) s'explique en remarquant que l'opérateur s'écrit

$$\begin{aligned} \mathsf{L} f &= \Delta_v f + (d+\gamma)\, \mathrm{div}_v\Big(\langle v \rangle^{-2} v f \Big) \\ &= \Delta_v f + (d+\gamma) \langle v \rangle^{-2} v \cdot \nabla_v f + (d+\gamma) \Big(d - 2 \tfrac{|v|^2}{\langle v \rangle^2} \Big) \langle v \rangle^{-2} f, \end{aligned}$$

[3] Si $\gamma + \beta = 2$, on obtient aussi une limite de diffusion classique, c'est dire $\alpha = 2$, mais on doit prendre pour changement d'échelle $\varepsilon^2 \ln(\varepsilon^{-1})$ à la place de ε^α dans (12), voir [161].

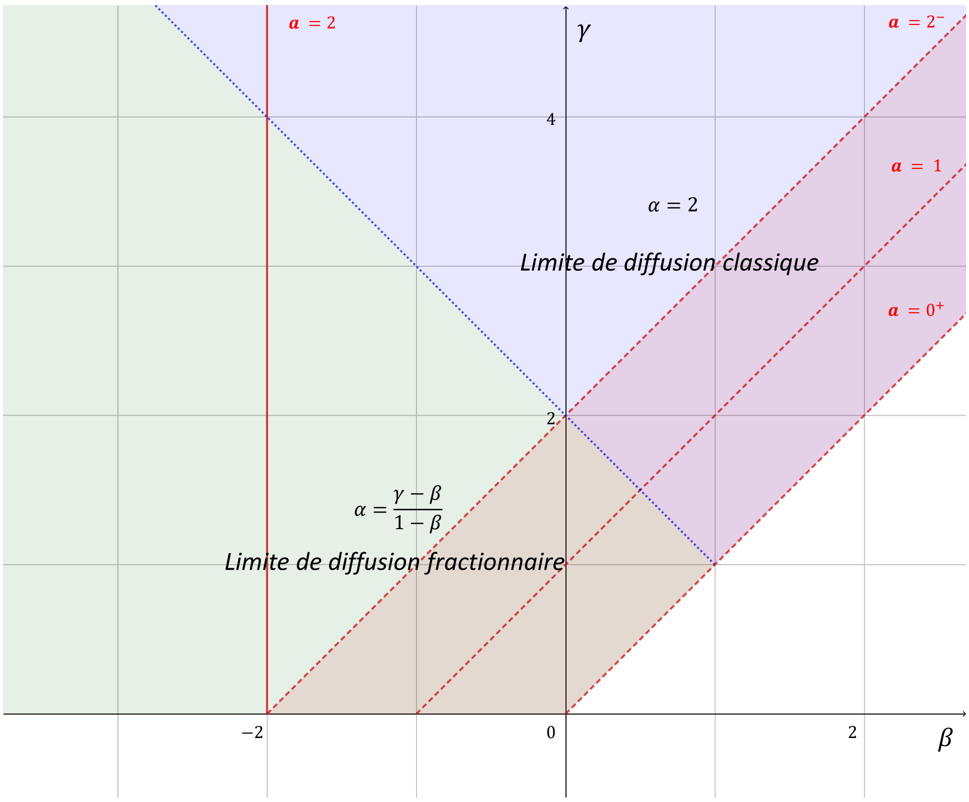

FIGURE 1 : Limite diffusive en fonction de β et γ pour les trois opérateurs considérés. La partie verte correspond au cas où $\alpha < 2$ et la partie bleue au cas où $\alpha = 2$. Le trait rouge vertical correspond au cas où on peut prendre pour L l'opérateur de Fokker-Planck alors que la zone rouge diagonale correspond à ceux où on peut prendre l'opérateur de Fokker-Planck fractionnaire.

et que $\nu(v)$ n'est rien d'autre que le terme d'ordre 0 de l'opérateur différentiel. Pour le cas de l'opérateur de Fokker-Planck fractionnaire (10) avec $\mathfrak{a} \in]0, 2[$, on montre au chapitre 5 que pour un état d'équilibre de la forme (14), on a

$$E_v(v) \underset{|v|\to\infty}{\simeq} \langle v \rangle^{\gamma-\mathfrak{a}}\, v.$$

C'est une des raisons pour laquelle on s'attend alors à avoir le même résultat de limite diffusive avec $\beta = \gamma - \mathfrak{a}$, ce qui revient à dire que l'on a alors

$$\begin{aligned} &\alpha = \frac{\mathfrak{a}}{\mathfrak{a} - (\gamma - 1)} && \text{si } 2\,(\gamma - 1) < \mathfrak{a} \\ &\alpha = 2 && \text{si } 2\,(\gamma - 1) \geq \mathfrak{a}. \end{aligned}$$

2.7 L'Équation de Keller-Segel

Dans le cadre de l'étude du mouvement de certaines bactéries, on observe des phénomènes de compétition entre la diffusion et l'agrégation due à la chimiotaxie, c'est-à-dire le fait que le mouvement de ces bactéries dépende de la concentration d'une substance chimique. E.F. Keller et L.A. Segel [131] modélisent le phénomène par l'équation suivante

$$\frac{\partial \rho}{\partial t} = \operatorname{div}(D_2 \nabla \rho) - \operatorname{div}(D_1 \nabla c)$$
$$\frac{\partial c}{\partial t} = D_c \, \Delta c - \frac{K_0 c}{1 + Kc} + \rho \, g(c),$$

où $\rho = \rho(t,x)$ et $c = c(t,x)$ sont respectivement la densité de bactéries et la concentration en chemo-attractant. Dans la première équation, la constante D_2 représente le coefficient de diffusion des bactéries, dû à leur mouvement en l'absence de la substance chimique, et D_1 est un coefficient qui indique à quel point les bactéries sont attirées par le chemo-attractant. Dans la deuxième équation, $g(c)$ est le taux de création de la substance attractive par les bactéries, qui est aussi supposé diffuser avec un taux D_c. Le terme contenant K_0 et K est un terme de perte dû à des réactions chimiques. En simplifiant un peu et en prenant en compte le fait que la substance chimique diffuse à un taux plus élevé que les bactéries et que D_1 est proportionnel à la concentration en bactéries, on se retrouve avec le **système de Keller-Segel parabolique-parabolique**

$$\frac{\partial \rho}{\partial t} = \Delta \rho - \lambda \operatorname{div}(\rho \nabla c)$$
$$\varepsilon \frac{\partial c}{\partial t} = \Delta c - \varepsilon c + \rho.$$

En considérant que la concentration c est en fait rapidement à l'équilibre, on peut faire tendre ε vers 0 et on obtient alors le **système de Keller-Segel parabolique-elliptique** qui s'écrit

$$\frac{\partial \rho}{\partial t} = \Delta \rho - \lambda \operatorname{div}(\rho \nabla c) \tag{15}$$
$$-\Delta c = \rho. \tag{16}$$

Ce modèle est très intéressant du point de vue mathématique puisqu'on peut montrer que l'on a une explosion en temps fini de la solution si le nombre de bactéries, c'est-à-dire la masse $M_0 = \int_{\mathbb{R}^d} \rho$, est trop grande. Ceci correspond au fait qu'une partie des bactéries s'agrègent en un unique point et implique l'apparition de solutions pour lesquelles la densité ρ n'est plus intégrable. Au contraire, si la masse n'est pas trop grande, les bactéries se répandent puisque leur dynamique est avant tout dirigée par la diffusion. Du point de vue biologique, on observe bien ce type de comportement.

Remarquons aussi que le système précédent peut se réécrire sous la forme d'une équation unique puisque dans $\mathbb{R}^d$, la deuxième équation se résout explicitement sous la forme

$c = K * \rho$ où K est le potentiel Coulombien $K(x) = \frac{C_d}{|x|^{d-2}}$ si $d \geq 3$ et $K(x) = -C_2 \ln(|x|)$ en dimension $d = 2$, ce qui mène à ce que l'on nommera l'**équation de Keller-Segel**

$$\frac{\partial \rho}{\partial t} = \Delta \rho + \lambda \, \mathrm{div}(E\rho) \,, \tag{17}$$

où $E = E(x) = -\nabla K * \rho$. Cette équation se retrouve aussi dans la description des systèmes d'étoiles en interaction gravitationnelle et sans collisions, et est connue parfois sous le nom d'équation de Chandrasekhar [68] ou système de Smoluchowski-Poisson (voir par exemple [35, 37]). Dans ce cas le phénomène d'explosion de la solution correspond à l'**effondrement** de l'étoile sous l'effet de sa propre masse ce qui est une justification simplifiée de la formation des trous noirs.

3 Liste des travaux présentés dans la thèse

Cette thèse est partagée en trois parties. La partie I présente les travaux concernant le lien entre échelle quantique et échelle mésoscopique, la partie II étudie le comportement en temps long à l'échelle mésoscopique et macroscopique et enfin la partie III étudie la définition et le comportement de modèles macroscopiques. Chacune de ces parties contient plusieurs chapitres qui correspondent chacun à un article.

- Chapitre 1 : article [133], accepté pour publication dans *Journal of Statistical Physics* sous réserve de modifications mineures.
- Chapitre 2 : article [134], *prépublication.*
- Chapitre 3 : article [132], soumis.
- Chapitres 4 et 5 : en préparation, articles en collaboration avec Émeric Bouin, Jean Dolbeault, Clément Mouhot et Christian Schmeiser.
- Chapitre 6 : article [135], en collaboration avec Samir Salem, accepté pour publication dans *Communications in Mathematical Sciences.* Les résultats sont aussi présentés en partie dans la note [190].
- Chapitre 7 : article [136], en collaboration avec Samir Salem, paru aux *Comptes-rendus de l'Académie des Sciences* (2019).

4 Principaux résultats obtenus

On présente dans cette section les résultats obtenus dans chacun des chapitres. Pour établir ceux-ci, on commence dans la plupart des cas par regarder la variation de certaines grandeurs physiques. Celles-ci sont souvent définies par des quantités intégrales puisqu'on regarde l'évolution de densités. On se place en particulier dans le cadre des espaces de Lebesgue défini pour $p \in [1, \infty]$ par

$$L^p := L^p(\mathbb{R}^d, \mathbb{R}) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{ mesurable, } \int_X |f|^p < \infty \right\} \text{ si } p < \infty$$
$$L^\infty := L^\infty(\mathbb{R}^d, \mathbb{R}) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{ mesurable, } \sup_X \operatorname{ess}(|f|) < \infty \right\},$$

avec l'identification usuelle des fonctions égales presque partout par passage au quotient. Comme on se place dans l'espace tout entier, on mesure aussi plus précisément les queues de distribution en utilisant des espaces à poids définis simplement de la manière suivante

$$\|f\|_{\mathcal{H}(m)} = \|fm\|_{\mathcal{H}} .$$

4.1 Limite semi-classique de l'équation de Hartree vers l'équation de Vlasov

Dans la partie I on s'intéresse à la question des liens entre l'équation de (Vlasov) et l'équation de (Hartree) présentées en section 2. L'objectif est de montrer que pour des potentiels singuliers tels que le potentiel Coulombien, la première équation donne une bonne approximation de la seconde lorsque $\hbar$ devient petit. Un premier résultat en ce sens avait été démontré par P.L. Lions et T. Paul [152], qui ont montré qu'il y avait, sous certaines conditions et à extraction d'une sous-suite près, convergence faible de la transformée de Wigner $f_\hbar$ de la solution $\boldsymbol{\rho}$ de l'équation de (Hartree) vers une solution f de l'équation de (Vlasov).

Dans notre cas, on démontre un résultat un peu différent puisqu'il apporte une estimation quantitative de la distance entre les solutions en fonction de $\hbar$. On doit se placer pour cela dans le cadre où il y a unicité de solutions fortes de l'équation de (Vlasov), c'est-à-dire f bornée et avec suffisamment de moments. On utilise une pseudo-distance introduite par F. Golse et T. Paul [105] qui se veut un équivalent de la distance de Wasserstein-Monge-Kantorovitch pour la mécanique quantique. Grâce à cette fonctionnelle, on peut en effet alors s'inspirer de la preuve de l'unicité de l'équation de (Vlasov) donnée par G. Loeper [157] au moyen de la distance de Wasserstein classique. Indiquons que ce type d'approche remonte à R. Dobrushin [79].

Propagation des moments et limite semi-classique

L'un des principaux résultats du Chapitre 1 est le fait que l'on peut obtenir des estimées indépendantes de $\hbar$ du même type que celles que l'on obtient pour l'équation de (Vlasov), et en particulier que pour de bonnes conditions initiales, la densité spatiale ρ est bornée. On introduit pour cela des normes de Schatten semi-classiques qui jouent le rôle des normes de Lebesgue cinétique en définissant

$$\|\boldsymbol{\rho}\|_{\mathcal{L}^p} = h^{-\frac{d}{p'}} \operatorname{Tr}(\boldsymbol{\rho}^p)^{\frac{1}{p}}, \tag{18}$$

où $p' = \frac{p}{p-1}$ désigne l'exposant conjugué de Hölder de p et Tr désigne la trace de l'opérateur compact positif $\boldsymbol{\rho}$. Pour simplifier, on se place en dimension $d = 3$ et dans le cas où le potentiel d'interaction s'écrit

$$K(x) = \frac{1}{|x|^a} \quad \text{pour un certain } a \in \left]-1, \frac{8}{7}\right[.$$

Le premier résultat concerne la propagation des moments en vitesse de la transformée de Wigner

$$\|\boldsymbol{\rho}\|_{\mathcal{L}^1(|\mathsf{p}|^n)} = \operatorname{Tr}(|\mathsf{p}|^n \boldsymbol{\rho}) = \iint_{\mathbb{R}^{2d}} f_\hbar(t,x,v)|v|^n \,\mathrm{d}x\,\mathrm{d}v.$$

et montre que ces moments sont propagés de façon globale dans certains cas, et au moins jusqu'à un certain temps maximal si le potentiel est très singulier.

Théorème 1. *Soit $\boldsymbol{\rho}$ une solution de l'équation de* (Hartree) *de condition initiale $\boldsymbol{\rho}^{\mathrm{in}} \in \mathcal{L}^1(|\mathsf{p}|^n) \cap \mathcal{L}^\infty$ pour un certain $n \in 2\mathbb{N}$. Alors il existe $T > 0$ tel que l'on a uniformément en $\hbar$*

$$\boldsymbol{\rho} \in L^\infty_{\mathrm{loc}}([0,T[, \mathcal{L}^1(|\mathsf{p}|^n))$$
$$\rho \in L^\infty_{\mathrm{loc}}([0,T[, L^1 \cap L^{1+\frac{n}{3}}).$$

De plus, on peut prendre $T = +\infty$ lorsque $a \leq \frac{4}{5}$ et l'on trouve alors des bornes à croissance polynomiale en temps, ou exponentielle dans le cas limite $a = \frac{4}{5}$.

Dans le cas de potentiels moins singuliers que le potentiel Coulombien, cette régularité sur ρ est suffisante pour obtenir l'unicité pour l'équation de (Vlasov) et une estimation quantitative de la limite semi-classique. Dans le cas du potentiel Coulombien cependant, on a besoin de $\rho \in L^\infty$. Cela peut s'obtenir pour l'équation de (Vlasov) en se servant du principe du maximum en considérant f bornée supérieurement par une fonction radiale décroissante intégrable, mais le principe du maximum n'est plus disponible dans le cas quantique. À la place, on regarde des normes de Lebesgue à poids et on fait tendre p vers l'infini.

Théorème 2. *Soit $\boldsymbol{\rho}$ une solution de l'équation de* (Hartree) *dont la condition initiale vérifie*

$$\boldsymbol{\rho}^{\mathrm{in}} \in \mathcal{L}^1(|\mathsf{p}|^{16}) \cap \mathcal{L}^\infty$$
$$\forall \mathrm{i} \in [\![1,d]\!],\ \mathsf{p}_\mathrm{i}^4 \boldsymbol{\rho}^{\mathrm{in}} \in \mathcal{L}^\infty,$$

indépendamment de $\hbar$, où $\mathsf{p}_\mathrm{i} := -i\hbar\partial_\mathrm{i}$. Alors il existe $T > 0$ tel que

$$\rho \in L^\infty_{\mathrm{loc}}([0,T[, L^\infty),$$

indépendamment de $\hbar$. Encore une fois, on peut prendre $T = +\infty$ si $a \leq \frac{4}{5}$.

Ces théorèmes sont les équivalents quantiques du théorème de P.L. Lions et B. Perthame [153] de propagation des moments pour l'équation de Vlasov-Poisson qui démontre aussi que sous les conditions suivantes sur la condition initale $f^{\text{in}} \in L^\infty_{x,\xi}(\mathbb{R}^6)$,

$$\iint_{\mathbb{R}^6} f^{\text{in}} |\xi|^{n_0} \, \mathrm{d}x \, \mathrm{d}\xi < C \text{ pour un certain } n_0 > 6, \tag{19}$$

et pour tout $R > 0$,

$$\sup_{(y,w)\in\mathbb{R}^6} \operatorname{ess} \{ f^{\text{in}}(y + t\xi, w), |x - y| \leq Rt^2, |\xi - w| \leq Rt \} \in L^\infty_{\text{loc}}(\mathbb{R}_+, L^\infty_x L^1_\xi). \tag{20}$$

alors il existe une unique solution f de condition initiale f^{in} à l'équation de (Vlasov) et qu'elle vérifie que la densité spatiale ρ est bornée.

Maintenant que l'on a de bonnes estimées indépendantes de $\hbar$, on obtient en particulier une certaine régularité du potentiel d'interaction $E = -\nabla K * \rho$ ce qui permet d'établir des estimées de stabilité semi-classique en utilisant les pseudo-distances de Wasserstein introduites par F. Golse et T. Paul [105]. Pour définir celles-ci, on introduit d'abord une notion de couplage semi-classique en écrivant que $\boldsymbol{\gamma} \in L^1(\mathbb{R}^{2d}, \mathcal{L}^1)$ est un couplage entre une densité de probabilité $f \in L^1(\mathbb{R}^{2d})$ et un opérateur densité positif $\boldsymbol{\rho} \in \mathcal{L}^1$ de trace 1, dénoté $\boldsymbol{\gamma} \in \mathcal{C}(f, \boldsymbol{\rho})$, si et seulement si

$$\operatorname{Tr}(\boldsymbol{\gamma}(z)) = f(z)$$
$$\int_{\mathbb{R}^{2d}} \boldsymbol{\gamma}(z) \, \mathrm{d}z = \boldsymbol{\rho}.$$

On peut alors définir la pseudo-distance de Wasserstein-Monge-Kantorovich semi-classique par

$$W_{2,\hbar}(f, \boldsymbol{\rho}) := \left(\inf_{\gamma \in \mathcal{C}(f,\boldsymbol{\rho})} \int_{\mathbb{R}^{2d}} \operatorname{Tr}(\mathbf{c}_\hbar(z) \boldsymbol{\gamma}(z)) \, \mathrm{d}z \right)^{\frac{1}{2}},$$

avec $\mathbf{c}_\hbar(z)\varphi(y) = (|x - y|^2 + |\xi - \mathsf{p}|^2)\, \varphi(y)$, $z = (x, \xi)$ et $\mathsf{p} = -i\hbar\nabla_y$. Ce n'est pas une distance puisque l'on a toujours $W_{2,\hbar}(f, \boldsymbol{\rho})^2 \geq d\hbar$ mais elle est tout de même comparable dans un certain sens à la distance de Wasserstein classique W_2 entre la densité cinétique classique et la transformée de Wigner de l'opérateur densité quantique

$$f_\hbar(x, v) = w_\hbar(\boldsymbol{\rho})(x, v) := \int_{\mathbb{R}^d} e^{-2i\pi y \cdot v} \varrho\left(x + \frac{hy}{2}, x - \frac{hy}{2} \right) \mathrm{d}y. \tag{21}$$

En effet, si $\boldsymbol{\rho} \in \mathcal{L}^1$ est un opérateur positif de trace 1 et $f \in L^1(\mathbb{R}^{2d})$ une densité de probabilité dont les deuxièmes moments en espace et en vitesse sont bornés, alors comme démontré dans [105], on a

$$W_2(f, \tilde{f}_\hbar)^2 \leq W_{2,\hbar}(f, \boldsymbol{\rho})^2 + d\hbar.$$

où $\tilde{f}_\hbar$ désigne la transformée de Husimi, qui est une fonction positive définie à partir de la transformée de Wigner par

$$\tilde{f}_\hbar = f_\hbar * G_h \quad \text{avec} \quad G_h(z) = h^{-d} \, e^{-|z|^2/h}.$$

De plus, le même type d'inégalité est vrai dans l'autre sens pour un cas particulier d'états qui sont des sommes états cohérents, comme expliqué en section 1.7.

Muni de cette fonctionnelle, on obtient alors le résultat suivant.

Théorème 3. *Soit* $a \in]-1/2, 1]$ *et soient* f *une densité de probabilité solution de l'équation de* (Vlasov) *et* $\boldsymbol{\rho}_\hbar$ *un opérateur positif de trace* 1 *solution de l'équation de* (Hartree)*. On suppose que* $\boldsymbol{\rho}$ *vérifie les hypothèses du Théorème* 2 *et que* f *a pour condition initiale*

$$f^{\text{in}} \in L^1 \cap L^\infty(\mathbb{R}^{2d}) \text{ vérifiant } (19) \text{ and } (20).$$

Alors il existe $T > 0$ *et une constante* C_T *dépendant des conditions initiales mais indépendante de* $\hbar$ *telle que*

$$W_{2,\hbar}(f(t), \boldsymbol{\rho}_\hbar(t)) \leq C_T \left(W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}_\hbar^{\text{in}}) + \sqrt{\hbar} \right),$$

Encore une fois, si $a \leq \frac{4}{5}$*, on peut prendre* $T = \infty$ *en remplaçant* C_T *par une fonction* C_t *dépendant du temps.*

Si $W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}_\hbar^{\text{in}}) > 1$ *et* $a = 1$*, il faut remplacer* $W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}_\hbar^{\text{in}})$ *par* $W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}_\hbar^{\text{in}})^{e^{T/\sqrt{2}}}$*.*

Dispersion semi-classique

Dans le chapitre 2, on va cette fois utiliser les propriétés de dispersion de l'équation de transport libre pour obtenir des estimées globales en temps pour des potentiels qui peuvent même être plus singuliers que le potentiel Coulombien. La dispersion est un effet dû au fait d'avoir initialement des vitesses très différentes dans un petit espace, ce qui en l'absence de collision fait que les particules vont avoir tendance à se *disperser*, c'est-à-dire à se séparer et occuper l'espace. Bien sûr, un des effets intéressants est que cela disperse les singularités de la distribution de probabilité, et on a donc des effets de gain d'intégrabilité et de régularisation. C'est l'effet à l'origine des lemmes de moyenne introduits par F. Golse, P.L. Lions, B. Pethame et R. Sentis [103] des inégalités de type Strichartz, et de manière plus générale d'un gain de compacité. Voir par exemple [18, 153, 179, 151].

Un exemple simple pour le comprendre est celui de l'équation de transport libre

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0,$$

de condition initiale $f(0, x, v) = f^{\text{in}}(x, v) \geq 0$. En effet, la solution est alors donnée explicitement par $f(t, x, v) = f^{\text{in}}(x - vt, v)$. En particulier, si on suppose que la distribution initiale est suffisamment localisée en espace dans le sens où il existe une fonction $g \in L^1(\mathbb{R}^d)$ telle que $f^{\text{in}}(x, v) \leq g(x)$ alors on obtient une borne uniforme sur la densité spatiale pour tous temps $t > 0$. En effet, on a

$$\rho(t, x) = \int_{\mathbb{R}^d} f^{\text{in}}(x - vt, v) \,\mathrm{d}v \leq \int_{\mathbb{R}^d} g(x - vt) \,\mathrm{d}v,$$

et donc, en effectuant le changement de variable $v = \frac{x-y}{t}$, on obtient

$$\|\rho(t, \cdot)\|_{L^\infty} \leq \frac{1}{t^d} \int_{\mathbb{R}^d} g(y) \,\mathrm{d}y = \frac{1}{t^d} \|g\|_{L^1}.$$

Par interpolation entre la norme L^1 et la norme L^∞ et en utilisant la conservation de la masse, on a alors pour tout $p \in [1, \infty]$

$$\|\rho(t,\cdot)\|_{L^p} \leq t^{-\frac{d}{p'}} \|g\|_{L^1} . \tag{22}$$

C'est ce type d'idées qui est déjà utilisé par C. Bardos et P. Degond [18] pour prouver l'existence globale pour le système de Vlasov-Poisson dans le cas de petites données initiales.

Pour le cas de l'équation de (Vlasov), on peut obtenir des résultats similaires en considérant des moments en espace, ce qui donne

$$\iint_{\mathbb{R}^{2d}} f(t,x,v)\,|x-vt|^n \,\mathrm{d}v\,\mathrm{d}x \in L^\infty(\mathbb{R}_+),$$

si $\int_{\mathbb{R}^d} \rho^{\mathrm{in}}(x)|x|^n \,\mathrm{d}x$ est suffisamment petit. On obtient des résultats similaires dans le cas de l'équation de (Hartree). Encore une fois, on se place pour simplifier dans le cas de la dimension 3 et on regarde un potentiel qui vérifie

$$|\nabla K| \leq \frac{1}{|x|^{a+1}} \mathbb{1}_{|x|\leq 1} + \frac{1}{|x|^{\tilde{a}+1}} \mathbb{1}_{|x|>1},$$

avec $a < \frac{8}{7}$ et $\tilde{a} > 1$. On doit en effet cette fois séparer le comportement local et le comportement à l'infini du potentiel puisque les interactions longues portées ont un impact très fort sur l'effet de la dispersion. De tels potentiels sont physiquement très réalistes pour des particules chargées puisque les effets d'écran diminuent l'effet du potentiel pour les grandes distances, comme c'est le cas par exemple pour le potentiel de (Yukawa). On montre alors le résultat suivant :

Théorème 4. *Soit $\boldsymbol{\rho}$ un opérateur positif solution de l'équation de* (Hartree) *dont la condition initiale vérifie pour un entier $n \geq 4$*

$$\boldsymbol{\rho}^{\mathrm{in}} \in \mathcal{L}^\infty \cap \mathcal{L}^1(|\mathsf{p}|^n),$$

indépendamment de $\hbar$. Alors il existe une constante $\mathcal{C} > 0$ dépendant de K et des conditions initiales telles que si

$$\int_{\mathbb{R}^d} \rho^{\mathrm{in}}(x)\,|x|^n \,\mathrm{d}x = \left\|\boldsymbol{\rho}^{\mathrm{in}}\right\|_{\mathcal{L}^1(|x|^n)} \leq \mathcal{C},$$

alors il existe une constante $C > 0$ telle que pour tout $p \in \left[1, 1+\frac{n}{3}\right]$ et pour tout $t > 0$

$$\mathrm{Tr}(|x - t\mathsf{p}|^n \boldsymbol{\rho}) \in L^\infty(\mathbb{R}_+)$$
$$\|\rho\|_{L^p} \leq C\, t^{-d/p'}.$$

De ce résultat, on peut à nouveau utiliser la stratégie du chapitre 1 pour obtenir des bornes, qui sont cette fois globales en temps, sur $\|\rho\|_{L^\infty}$ et obtenir un résultat similaire de limite semi-classique pour $a \leq 1$.

Formalisme cinétique en mécanique quantique

La stratégie consiste finalement avant tout à mettre en parallèle les formalismes quantique et cinétique. On considère un opérateur densité $\boldsymbol{\rho}$ défini par son noyau

$$\boldsymbol{\rho}\,\varphi = \int_{\mathbb{R}^d} \varrho(x,y)\varphi(y)\,\mathrm{d}y.$$

Dans ce cas, on voit que la densité spatiale définie par l'équation (5) s'écrit formellement

$$\rho(x) = \operatorname{diag}(\boldsymbol{\rho}) := \varrho(x,x). \tag{23}$$

Un outil bien connu pour obtenir une fonction de l'espace des phases qui se comporte bien dans la limite semi-classique à partir de la modélisation quantique est la transformée de Wigner $f_\hbar(x,v) = w_\hbar(\boldsymbol{\rho})(x,v)$ qui est définie par l'équation (21). Une telle fonctionnelle vérifie en particulier $\int_{\mathbb{R}^d} f_\hbar\,\mathrm{d}v = \rho$ et plus généralement pour toute observable $A(x)$ ou $B(v)$, on peut calculer sa valeur moyenne en écrivant

$$\operatorname{Tr}(A(x)\boldsymbol{\rho}) = \iint_{\mathbb{R}^{2d}} A(x) f_\hbar(x,v)\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^d} A(x)\rho(x)\,\mathrm{d}x \tag{24}$$

$$\operatorname{Tr}(B(\mathsf{p})\boldsymbol{\rho}) = \iint_{\mathbb{R}^{2d}} B(v) f_\hbar(x,v)\,\mathrm{d}x\,\mathrm{d}v. \tag{25}$$

De plus, si $\boldsymbol{\rho}$ vérifie l'équation de (Hartree) sans potentiel, c'est-à-dire avec $V = 0$, alors $f_\hbar$ vérifie l'équation de transport libre

$$\frac{\partial f_\hbar}{\partial t} + v\cdot\nabla_x f_\hbar = 0,$$

et si plus généralement $V = V(x)$, on peut montrer que $f_\hbar$ vérifie une équation qui s'écrit sous la forme

$$\frac{\partial f_\hbar}{\partial t} + v\cdot\nabla_x f_\hbar + \mathcal{K}_\hbar * f_\hbar = 0,$$

où, comme démontré par P.L. Lions et T. Paul [152], formellement ou sous de bonnes conditions, on a $\mathcal{K}_\hbar * f \underset{\hbar\to 0}{\to} E(x)\cdot\nabla_v f$ avec $E = -\nabla V$. Cependant, bien que $f_\hbar$ est une quantité réelle, elle n'est pas toujours positive ce qui fait que ce n'est pas une densité de probabilité et pose des difficultés pour obtenir les mêmes estimées que pour l'équation de (Vlasov).

La stratégie est alors plutôt de considérer l'analogie formelle entre l'équation de (Hartree) vérifiée par $\boldsymbol{\rho}$ et l'équation de (Vlasov) vérifiée par la densité cinétique f et qui s'écrit grâce aux crochets de Poisson

$$\frac{\partial f}{\partial t} = \{H_f, f\} \text{ avec } H_f = \frac{|v|^2}{2} + V(x).$$

En effet, **$\boldsymbol{\rho}$ est bien positif**, même si c'est cette fois au sens des opérateurs. L'intégrale sur l'espace des phases correspond alors à la trace d'après les équations (24) et (25), et même plus précisément l'intégrale en vitesse correspond à la diagonale comme défini par (23). Pour les normes de Lebesgue, on peut déjà remarquer que l'on a par le Théorème de Plancherel l'identité suivante

$$\|f_\hbar\|_{L^2(\mathbb{R}^{2d})} = h^{-\frac{d}{2}} \|\varrho\|_{L^2(\mathbb{R}^{2d})} = h^{-\frac{d}{2}} \operatorname{Tr}\left(\boldsymbol{\rho}^2\right)^{\frac{1}{2}}.$$

Ensuite, on a les inégalités d'interpolation classiques dans l'étude des équations cinétiques pour $f \geq 0$

$$\|\rho\|_{L^p} \leq \|f\|_{L^r(\mathbb{R}^{2d})}^{\frac{r'}{p'}} \left(\iint_{\mathbb{R}^{2d}} f(x,v)|v|^n \,\mathrm{d}x\,\mathrm{d}v\right)^{1-\frac{r'}{p'}}, \tag{26}$$

où $p' = r' + \frac{d}{n}$. Or, des inégalités semblables ont lieu au niveau quantique, qui étaient déjà connues pour $n = 2$ sous le nom d'inégalités de Lieb-Thirring, sous la forme

$$\|\rho\|_{L^p} \leq h^{-\frac{d}{p'}} \operatorname{Tr}(\boldsymbol{\rho}^r)^{\frac{r'}{rp'}} \operatorname{Tr}(|\mathsf{p}|^n \boldsymbol{\rho})^{1-\frac{r'}{p'}}.$$

Ces inégalités se réécrivent exactement sous la forme (26) pour la transformée de Wigner $f_\hbar$ bien que celle-ci ne soit pas positive. Ceci suggère la définition (18) des normes de Lebesgue cinétiques quantiques.

On peut donc écrire le tableau suivant des correspondances entre le monde quantique et le monde classique :

Modèles cinétiques	Modèles quantiques
$f = f(x,v)$	$\boldsymbol{\rho} = \sum_{\mathrm{j}\in\mathbb{N}} \lambda_{\mathrm{j}} \|\psi_{\mathrm{j}}\rangle\langle\psi_{\mathrm{j}}\|$
$z = (x,v)$	$(x,\mathsf{p}) = (x, -i\hbar\nabla)$
$H = \frac{\|v\|^2}{2} + V(x)$	$\mathsf{H} = \frac{\|\mathsf{p}\|^2}{2} + V(x) = \frac{-\hbar^2}{2}\Delta + V(x)$
$\rho = \int_{\mathbb{R}^d} f \,\mathrm{d}v$	$\operatorname{diag}(\boldsymbol{\rho}) = \sum_{\mathrm{j}\in\mathbb{N}} \lambda_{\mathrm{j}} \|\psi_{\mathrm{j}}\|^2$
$j_f = \rho\, u = \int_{\mathbb{R}^d} f v \,\mathrm{d}v$	$j_{\boldsymbol{\rho}} = \frac{1}{2}\operatorname{diag}(\boldsymbol{\rho}\mathsf{p} + \mathsf{p}\boldsymbol{\rho}) = \hbar \sum_{\mathrm{j}\in\mathbb{N}} \lambda_{\mathrm{j}} \Im(\overline{\psi_{\mathrm{j}}}\nabla\psi_{\mathrm{j}})$
$M_0 = \iint_{\mathbb{R}^{2d}} f \,\mathrm{d}x\,\mathrm{d}v$	$\operatorname{Tr}(\boldsymbol{\rho}) = \sum_{\mathrm{j}\in\mathbb{N}} \lambda_{\mathrm{j}}$
$M_n = \iint_{\mathbb{R}^{2d}} f\|v\|^n \,\mathrm{d}x\,\mathrm{d}v$	$\operatorname{Tr}(\|\mathsf{p}\|^n \boldsymbol{\rho}) = \hbar^n \sum_{\mathrm{j}\in\mathbb{N}} \lambda_{\mathrm{j}} \int_{\mathbb{R}^d} \|\nabla^{\frac{n}{2}}\psi_{\mathrm{j}}\|^2$
$\|f\|_{L^p(\mathbb{R}^{2d})}$	$\|\boldsymbol{\rho}\|_{\mathcal{L}^p} = h^{-\frac{d}{p'}} \operatorname{Tr}(\boldsymbol{\rho}^p)^{\frac{1}{p}}$
$\|f\|_{L^\infty(\mathbb{R}^{2d})}$	$\|\boldsymbol{\rho}\|_{\mathcal{L}^\infty} = h^{-d} \|\boldsymbol{\rho}\|_{\mathscr{B}}$.

On a désigné par $\Im$ la partie imaginaire. De même en regardant les transformées de Wigner on peut définir les dérivées en x et v de manière suivante

$$\nabla_x f = \{v, f\} \qquad \eth_x \boldsymbol{\rho} = [\nabla, \boldsymbol{\rho}]$$

$$\nabla_v f = \{x, f\} \qquad \eth_v \boldsymbol{\rho} = \left[\frac{x}{i\hbar}, \boldsymbol{\rho}\right],$$

et les espaces de Sobolev associés en posant

$$|f|^p_{W^{1,p}} = \|\nabla_x f\|^p_{L^p} + \|\nabla_v f\|^p_{L^p} \qquad |\boldsymbol{\rho}|^p_{\mathcal{W}^{1,p}} = \|\eth_x \boldsymbol{\rho}\|^p_{\mathcal{L}^p} + \|\eth_v \boldsymbol{\rho}\|^p_{\mathcal{L}^p}.$$

La plus grosse différence reste le fait que les opérateurs ne commutent pas forcément en mécanique quantique. En passant du modèle des transformées de Wigner à celui des opérateurs densité, on a donc gagné la positivité de $\boldsymbol{\rho}$ mais perdu la commutativité qui était belle et bien présente pour les transformées de Wigner et les observables en x et v.

4.2 Comportement asymptotique de modèles cinétiques

Dans la partie II, on s'intéresse au comportement en temps long d'équations cinétiques avec un terme de collision linéaire L qui induit l'existence d'un **équilibre local** $F(v)$ au niveau microscopique. Cet équilibre est défini comme étant l'équilibre dans le cas où la distribution de particules dans l'espace des phases est homogène en espace, c'est-à-dire ne dépend pas de la variable x. Dans le cas où les températures ne sont pas trop élevées, on observe en effet que l'équilibre thermique s'effectue plus rapidement que la dissipation des particules dans l'espace et on peut donc considérer qu'en temps long on obtient une vitesse de dissipation du même ordre que si les particules étaient depuis le début à l'équilibre thermique. Ce type de situation correspond justement au cas des limites de diffusion décrit en section 2.6.

On commence par regarder dans le chapitre 3 le cas homogène en espace, ce qui permet de comprendre la phase de mise en équilibre des vitesses. La densité de particules $f = f(t, v)$ ne dépend alors plus de la variable spatiale x et vérifie

$$\frac{\partial f}{\partial t} = \mathsf{L} f. \tag{27}$$

Dans ce cas, l'équilibre local n'est rien d'autre que l'état stationnaire de l'équation (27). On cherche alors à mesurer la vitesse de convergence vers cet état stationnaire ce qui nous donnera une idée du fonctionnement de l'étape d'équilibre thermique rapide dans le cas non-homogène. La théorie des semi-groupes permet en particulier d'écrire la solution d'une telle équation de condition initiale f^{in} sous la forme

$$f(t, v) = e^{t\mathsf{L}} f^{\mathrm{in}}(v),$$

et l'on comprend que ce problème est aussi fortement lié à l'étude spectrale de l'opérateur L. En effet, si l'opérateur $-\mathsf{L}$ est positif et s'il admet une plus petite valeur propre λ_0 dans un certain espace $\mathcal{H}$, alors dans cet espace on aura

$$\|f\|_{\mathcal{H}} \leq e^{-\lambda_0 t} \left\| f^{\mathrm{in}} \right\|_{\mathcal{H}}.$$

Cependant, un tel résultat avec $\lambda_0 > 0$ n'est pas possible s'il y a un état stationnaire, puisque ce dernier est un vecteur propre pour la valeur propre 0. Pour obtenir une convergence rapide vers l'état stationnaire, on a alors envie de regarder la première valeur propre non nulle $\lambda_1 > 0$, c'est-à-dire que l'on se place dans l'orthogonal à l'espace engendré par F dans $\mathcal{H}$. C'est la distance entre ces deux valeurs propres que l'on appelle le **trou spectral**. L'existence d'un tel trou implique donc une convergence exponentielle vers l'équilibre

$$\|f - F\|_{\mathcal{H}} \leq e^{-\lambda_1 t} \left\| f^{\text{in}} - F \right\|_{\mathcal{H}}.$$

Dans le cas où L est l'opérateur de Fokker-Planck (8) avec un équilibre de type exponentiel $F(v) = e^{-|v|^\gamma}$ pour $\gamma > 0$, ce problème a déjà été beaucoup étudié (voir par exemple [149, 184, 74, 123, 15, 16, 84, 42, 81, 167, 130, 174, 169]). Il s'avère en particulier qu'il y a un gap spectral dans des espaces à poids exponentiels du type $L^p(e^{|v|^k})$ lorsque $\gamma > 1$ et dans des espaces à poids polynomiaux du type $L^p(\langle v \rangle^k)$ lorsque $\gamma > 2$, où $\langle v \rangle = \sqrt{1 + |v|^2}$.

Lorsqu'il n'y a plus de gap spectral mais qu'il y a toujours un état stationnaire, on peut alors parfois obtenir un taux de convergence vers l'état d'équilibre. Cependant l'espace de départ et d'arrivée seront différents et le taux ne sera généralement plus exponentiel.

L'équation de Fokker-Planck fractionnaire

On étudie dans le chapitre 3 le cas où le L est un opérateur de Fokker-Planck fractionnaire, c'est-à-dire l'équation

$$\frac{\partial f}{\partial t} = \mathsf{L} f := \Delta_v^{\frac{\mathfrak{a}}{2}} f + \operatorname{div}_v(E(v) f). \tag{FFP}$$

Remarquons que comme expliqué en section 2.6, cette équation peut aussi être écrite en remplaçant v par x puisqu'elle peut être vue comme la limite de diffusion d'équations cinétiques avec une force $E = E(x)$ dépendant uniquement de la variable d'espace. On prend ici des potentiels dont le modèle est

$$E(v) = \langle v \rangle^\beta v = \nabla \left(\frac{\langle v \rangle^{\beta+2}}{\beta + 2} \right).$$

Ils sont du même type que ceux considérés pour l'équation de Fokker-Planck classique pour laquelle ils correspondent aux états stationnaires de la forme $F(v) = e^{-\frac{\langle v \rangle^\gamma}{\gamma}}$ avec $\gamma = \beta + 2$. Cependant, une des difficultés ici est qu'on ne peut pas calculer explicitement ces états stationnaires. On peut néanmoins en connaître des propriétés puisque ce sont des solutions de l'équation (FFP).

Une autre difficulté provient du caractère non-local de l'opérateur $\Delta_v^{\frac{\mathfrak{a}}{2}}$. En effet, on ne peut par exemple pas calculer le Laplacien fractionnaire d'une fonction poids de type exponentiel, puisque l'on peut définir le Laplacien fractionnaire uniquement pour des fonctions qui croissent à l'infini au maximum comme $|v|^k$ avec $k < \mathfrak{a}$.

Enfin, une difficulté supplémentaire s'ajoute lorsque $\mathfrak{a} < 1$, puisque dans ce cas l'ordre du Laplacien fractionnaire comme opérateur différentiel est inférieur à celui de l'opérateur divergence qui apparaît dans le terme de transport, et cela fait que ce dernier ne peut plus être traité comme une perturbation et l'on perd une partie des propriétés de régularisation que l'on a dans le cas $\mathfrak{a} > 1$. En particulier, le fait de savoir si la solution d'une telle équation devient immédiatement bornée uniformément en espace reste un problème ouvert lorsque $\beta > 0$. De même, pour β assez grand, on n'est même plus sûr que la solution est dans L^2 et l'on se place donc dans des espaces de type $L^p(\langle v\rangle^k)$ avec $p > 1$ et $k \in]0, \mathfrak{a}[$.

Cependant, le caractère non-local offre une autre propriété intéressante qui est de pouvoir assez facilement obtenir des bornes inférieures sur les solutions. La technique consiste à isoler les grands sauts dans l'écriture intégrale du Laplacien fractionnaire en écrivant

$$\frac{1}{|v|^{d+\mathfrak{a}}} = \kappa(v) + \kappa^c(v) := \frac{\mathbb{1}_{|v|\leq R}}{|v|^{d+\mathfrak{a}}} + \frac{\mathbb{1}_{|v|>R}}{|v|^{d+\mathfrak{a}}},$$

de telle sorte qu'on obtient alors

$$\Delta_v^{\frac{\mathfrak{a}}{2}} f = \Delta_{v,R}^{\frac{\mathfrak{a}}{2}} f + \kappa^c * f - \|\kappa^c\|_{L^1} f.$$

On peut donc découper l'opérateur L en deux bouts $\mathsf{L} = \mathsf{A} + \mathsf{B}$ où $\mathsf{A}f = \kappa^c * f$. Comme $e^{t\mathsf{B}}$ est un opérateur positif, on peut utiliser la formule de Duhamel qui donne

$$e^{t\mathsf{L}} = e^{t\mathsf{B}} + e^{t\mathsf{B}} \star \mathsf{A}\, e^{t\mathsf{L}} \geq e^{t\mathsf{B}} \star \mathsf{A}\, e^{t\mathsf{L}}.$$

On se sert ensuite du fait que $e^{t\mathsf{L}}$ conserve le fait d'avoir de la masse dans une boule, que A transforme la masse dans une boule en une borne inférieure par $C\langle v\rangle^{-(d+\mathfrak{a})}$ et que $e^{t\mathsf{B}}$ conserve le fait d'être au-dessus d'une fonction $C\langle v\rangle^{-(d+\gamma)}$ avec $\gamma > \mathfrak{a} + \beta_+$. On obtient à la fin que pour toute solution f de condition initiale $f^{\text{in}} \in L^1 \cap L^p(m)$ positive et pour tout $R > 0$, il existe $\lambda > 0$ tel que pour tout $(t, v) \in \mathbb{R}_+ \times \mathbb{R}^d$

$$f(t,v) \geq C\, \frac{te^{-\lambda t}}{\langle v\rangle^{d+\gamma}} \int_{B_R} f^{\text{in}}\, \mathrm{d}v. \tag{28}$$

Pour prouver la convergence vers l'équilibre, il y a deux stratégies différentes. Si $\beta \leq 0$, alors on sait que l'état d'équilibre est borné et l'on peut donc se servir d'une inégalité de Poincaré fractionnaire locale et de la dissipation de la norme $L^p(\langle v\rangle^k)$ pour obtenir finalement pour un certain $p > 1$ et une certaine constante $b > 0$ l'inégalité suivante

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d} |f|^p m_\lambda^p\, \mathrm{d}v\right) \leq -b \int_{\mathbb{R}^d} |f|^p m_\lambda^p \langle v\rangle^\beta\, \mathrm{d}v,$$

où on a pris une combinaison du poids $m(x) = \langle x\rangle^k$ et de l'état d'équilibre sous la forme

$$m_\lambda(v)^p = m(v)^p + \lambda F(v)^{1-p} \underset{|v|\to\infty}{\sim} C\langle v\rangle^{kp}.$$

Comme le coefficient β est négatif cependant, on ne peut pas en déduire une convergence exponentielle, mais on est obligé d'utiliser l'inégalité de Hölder, ce qui est à l'origine du fait que l'on obtient un taux de convergence polynomial dans ce cas.

Si $\beta > 0$, on ne sait pas prouver que F est bornée et cela semble empêcher aussi de prouver une inégalité de Poincaré fractionnaire locale. On utilise alors des techniques inspirées des probabilités, introduites par Harris puis développées par S.P. Meyn et R.L. Tweedie [162] puis M. Hairer et J.C. Mattingly [112]. Pour cela, on se place du point de vue dual dans lequel

$$\mathsf{P}_t := e^{t\mathsf{L}^*}$$

est un semi-groupe de Markov qui est continu en temps à valeur dans $L^\infty(\langle v\rangle^{-k})$. Les ingrédients sont alors les suivants. On prouve une inégalité de **Foster-Lyapunov** qui joue le rôle de la dissipation de la norme $L^1(m)$ dans la stratégie précédente et qui s'écrit pour $m(v) = \langle v\rangle^k$

$$\mathsf{P}_t m \leq \varepsilon m + c,$$

avec $\varepsilon \in]0,1[$. Le deuxième ingrédient est de prouver un résultat de **positivité** sous la forme

$$\mathsf{P}_t \geq \langle \nu_t, \cdot\rangle \, \mathbb{1}_{m(v)<r},$$

pour une certaine fonction positive ν_t. Or ceci est exactement l'écriture duale de l'inégalité (28) pour $r = m(R)$ et $\nu_t(x) = \frac{Cte^{-\lambda t}}{\langle v\rangle^{d+\gamma}}$. Cette propriété remplace d'une certaine manière l'utilisation d'une inégalité de Poincaré locale, mais ne nécessite pas de connaître l'état d'équilibre ! En combinant ces deux ingrédients, comme montré dans [112], on obtient que P_t est une contraction dans une semi-norme qui s'écrit

$$|\varphi|_{\dot L^\infty(m_\lambda^{-1})} := \sup_{(v,v_*)\in\mathbb{R}^{2d}} \left(\frac{|\varphi(v)-\varphi(v_*)|}{m_\lambda(v)+m_\lambda(v_*)}\right) = \inf_{c\in\mathbb{R}} \|\varphi - c\|_{L^\infty(m_\lambda^{-1})},$$

où cette fois, $m_\lambda(v) = 1 + \lambda m(v)$. De ceci on déduit la convergence exponentielle dans $L^1(m)$ et par des propriétés de régularisation de $L^1(m)$ dans $L^p(m)$, on déduit un résultat similaire dans des normes $L^p(m)$ avec $p > 1$. Le résultat s'écrit finalement ainsi.

Théorème 5. *Soit $\beta > -\mathfrak{a}$ et $m := \langle v\rangle^k$ pour $0 \leq k < (\mathfrak{a}\wedge 1)$. Alors, si $\beta \geq 0$, il existe $p_\beta > 1$ et $a > 0$ tel que si $p \in [1, p_\beta)$,*

$$\|f - F\|_{L^p(m)} \lesssim e^{-at} \left\|f^{\rm in} - F\right\|_{L^p(m)}.$$

Si $\beta \in]-\mathfrak{a}, 0[$, alors il existe $p^* > 1$ tel que pour tout $p \in [1, p^*[$ et $\bar k < k$, on a

$$\|f - F\|_{L^p(\bar m)} \lesssim \langle t\rangle^{-a} \left\|f^{\rm in} - F\right\|_{L^p(m)},$$

où $\bar m = \langle v\rangle^{\bar k}$ et $a = \frac{\bar k - k}{|\beta|}$ si $p > 1$. Lorsque $p = 1$, a peut être n'importe quel nombre réel satisfaisant $a < \frac{\bar k - k}{|\beta|}$.

Équations cinétiques : hypocoercivité

Dans les chapitres 4 et 5, on regarde cette fois le comportement en temps long d'équations cinétiques qui dépendent à la fois de x et v et qui s'écrivent

$$\frac{\partial f}{\partial t} + \mathsf{T} f = \mathsf{L} f, \qquad (29)$$

où T est un opérateur de transport et L un terme de collision linéaire. L'objectif est d'étudier l'effet de l'opérateur L sur la variable d'espace alors qu'il agit au premier ordre sur la variable de vitesse. C'est dans cette optique que s'est développée la théorie de l'hypocoercivité qui consiste à trouver des normes qui mixent les variables x et v pour obtenir des taux de décroissance exponentiels vers l'équilibre du semi-groupe associé à ce type d'équation.

Dans notre cas, on regarde le cas où $\mathsf{T} = v \cdot \nabla_x$ et il n'y a donc pas confinement spatial. En supposant que la convergence vers l'équilibre thermodynamique est rapide, c'est-à-dire que $f \simeq \rho(t,x)F(v)$, on voit que l'on s'attend à retrouver la dynamique de l'équation obtenue en prenant la limite de diffusion comme décrit en section 2.6. La stratégie consiste donc à découper la solution en une **partie macroscopique** et une **partie microscopique**

$$f = \rho F + (f - \rho F),$$

et à dire que le second terme va vite devenir négligeable alors que le premier suivra la dynamique macroscopique. Une façon de capturer cet effet, introduite par J. Dolbeault, C. Mouhot et C. Schmeiser dans [81], est de construire une **entropie**, ou plus précisément une **fonctionnelle de Lyapunov**, qui fasse apparaître la coercivité macroscopique et microscopique tout en gardant bornés et petits les termes supplémentaires. On se place pour cela dans l'espace de Hilbert $\mathcal{H} = L^2_{x,v}(F^{-1/2})$ muni du produit scalaire

$$\langle f, g \rangle_{\mathcal{H}} = \iint_{\mathbb{R}^{2d}} f\, g\, F^{-1} \,\mathrm{d}x\,\mathrm{d}v.$$

Cette entropie s'écrit alors sous la forme

$$\mathsf{H}(f) = \|f\|_{\mathcal{H}} + \delta \langle \mathsf{A} f, f \rangle_{\mathcal{H}}, \tag{30}$$

où le premier terme capture la coercivité microscopique et le deuxième terme est défini par

$$\mathsf{A} = \left(1 + |\mathsf{T}\Pi|^2\right)^{-1} (\mathsf{T}\Pi)^*, \tag{31}$$

où l'on a noté Π l'opérateur de projection orthogonale sur F dans $\mathcal{H}$, c'est-à-dire $\Pi f = \rho(x)F(v)$. Elle est équivalente à la norme $\mathcal{H}$ pour δ assez petit. Comme montré dans [44], lorsque L est l'opérateur de Fokker-Planck (8) ou l'opérateur de Boltzmann linéaire (13) et qu'il a un trou spectral, on retrouve bien que la solution décroit dans $\mathcal{H}$ à un taux $t^{-d/4}$ si elle est initialement dans $\mathcal{H} \cap L^1(\mathbb{R}^{2d})$, ce qui correspond au taux de l'équation de la chaleur.

Au chapitre 4, on regarde cependant le cas où il n'y a plus de gap spectral mais que l'équilibre local a toujours tous ses moments bornés, ce qui est le cas lorsque

$$F(v) = C_\gamma e^{-\langle v \rangle^\gamma}.$$

pour $\gamma \in]0,1[$ et implique que la limite de diffusion est classique et non fractionnaire. L'inégalité de Poincaré qui était associée à la dissipation de la norme $\mathcal{H}$ est alors remplacée par une inégalité de Poincaré à poids de la forme

$$- \langle \mathsf{L} f, f \rangle_{L^2_{x,v}(F^{-1/2})} \leq \mathcal{C}_m \|(1-\Pi) f\|_{L^2_{x,v}\left(\langle v \rangle^{\beta/2} F^{-1/2}\right)}, \tag{32}$$

avec $\beta < 0$. Pour compenser cette perte de poids, on prend une donnée initiale avec suffisamment de moments initialement et on obtient le résultat suivant en posant $\beta = 2(1-\gamma)$ si L est l'opérateur de Fokker-Planck (8), et sous certaines conditions sur L si L est l'opérateur de Boltzmann linéaire défini par (13).

Théorème 6. *Soient $\gamma \in]0,1[$ et $k > 0$. Alors il existe une constante $C > 0$ tel que pour toute solution f de (29) de condition initiale $f^{\mathrm{in}} \in \mathcal{H}(\langle v \rangle^{k/2}) \cap L^1(\mathbb{R}^{2d})$ on a*

$$\|f(t,\cdot,\cdot)\|^2_{L^2_{x,v}(F^{-1/2})} \leq C \, \frac{\|f^{\mathrm{in}}\|^2_{L^2_{x,v}(\langle v \rangle^{k/2} F^{-1/2})} + \|f^{\mathrm{in}}\|^2_{L^1(\mathbb{R}^{2d})}}{(1+t)^a},$$

où $a = \min\{d/2, k/|\beta|\}$.

On voit en particulier que si la condition initiale n'a pas assez de moments bornés en vitesse, alors le taux se dégrade et que ceci est dû au fait que la convergence vers l'équilibre en vitesse devient plus lente que la dispersion dans l'espace. Cependant, dès qu'il y a suffisamment de moments initiaux, on arrive toujours à obtenir le taux de décroissance de l'équation de la chaleur macroscopique.

Au chapitre 5, on regarde cette fois le cas où l'état d'équilibre des vitesses s'écrit sous la forme

$$F(v) = \frac{C_\gamma}{\langle v \rangle^{d+\gamma}},$$

auquel cas la limite de diffusion peut être fractionnaire et on reprend donc les notations de la section 2.6. La méthode semble pouvoir s'appliquer pour de nombreux opérateurs, et on traite ici le cas des trois opérateurs dont on connait une limite fractionnaire, c'est à dire l'opérateur de Fokker-Planck, de Boltzmann linéaire et de Fokker-Planck fractionnaire.

Si $\beta \leq 0$, on a à nouveau une perte de coercivité de l'opérateur L. Cependant, le problème principal consiste à définir un nouvel opérateur A qui soit compatible avec la limite de diffusion fractionnaire et qui soit bien défini même si l'état d'équilibre a peu de moments finis. En s'inspirant du A défini ci-dessus par (31) et du symbole obtenu dans le passage à la limite diffusive fractionnaire dans [161], on définit A par son symbole en prenant la transformée de Fourier dans la variable x, c'est-à-dire $\mathsf{A}f = \mathcal{F}_\xi\Big(\mathsf{A}_\xi \, \widehat{f}(\xi,v)\Big)$ et on définit A_ξ par

$$\mathsf{A}_\xi = \frac{C_\xi}{1+\langle v \rangle^2 |\xi|^2} \, \mathsf{\Pi T} \, \frac{\langle v \rangle^{-\beta}}{1+\langle v \rangle^{2|1-\beta|} |\xi|^2}.$$

Comme précédemment, on utilise une inégalité de Poincaré à poids du type (32) pour tirer de l'information de la partie microscopique de la dissipation de l'entropie, et l'inégalité de Nash qui donnait la coercivité au niveau macroscopique est remplacée par une inégalité de Nash fractionnaire. On obtient le résultat suivant

Théorème 7. *Soient $\gamma > |\beta|$ et $k \in]0,\gamma[$ si $\beta < 0$ ou $k = 0$ si $\beta > 0$. Soit f une solution de (29) pour un opérateur L défini par (8), (13) ou (10) et de condition initiale*

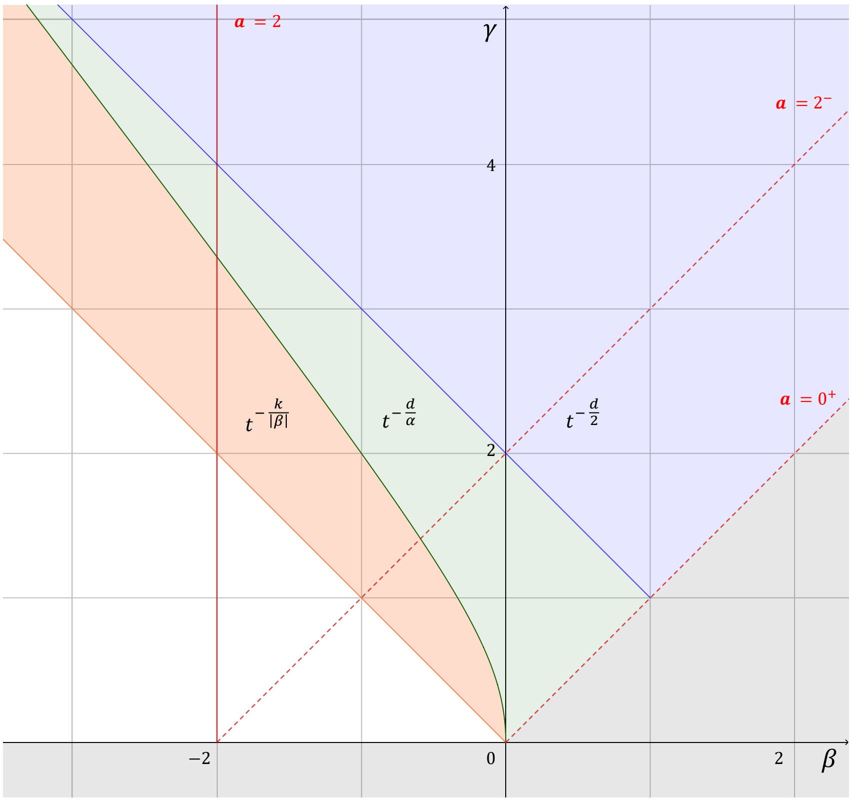

FIGURE 2 : Taux de décroissance du carré de la norme L^2 à poids obtenus au chapitre 5 si $d = 3$ et k est proche de γ. La partie bleue correspond au cas où la décroissance est la même que l'équation de la chaleur, la partie verte à l'équation de la chaleur fractionnaire, et la partie orange au taux de l'équilibre thermodynamique. Dans la partie blanche, l'inégalité de Poincaré (32) ne fonctionne plus.

$f^{\text{in}} \in L^1_{x,v} \cap L^2_{x,v}(\langle v \rangle^{k/2} F^{-1/2})$. *Alors, sous certaines conditions sur* L, *on obtient*

$$\|f\|^2_{L^2_{x,v}(F^{-1/2})} \leq C \, \frac{\|f^{\text{in}}\|^2_{L^1_{x,v}} + \|f^{\text{in}}\|^2_{L^2_{x,v}(\langle v \rangle^{k/2} F^{-1/2})}}{(1+t)^{\min\left(\frac{d}{\alpha}, \frac{k}{|\beta|}\right)}},$$

où α *est tel que défini en section 2.6.*

Les résultats sont résumés dans la Figure 2. Physiquement, plus γ sera petit, plus il y aura des particules avec des grandes vitesses, et plus β est grand, plus les grandes vitesses seront freinées par la force de friction. C'est ce qui explique que pour β ou γ petit, on obtient une limite diffusive fractionnaire puisque les grandes vitesses ne sont pas freinées, et sont vues comme des sauts après changement d'échelle. Et dans cette situation, on comprend aussi pourquoi l'équilibre des vitesses est lent.

4.3 Comportement d'équations de type Keller-Segel

Dans la partie III, on regarde le comportement de modèles macroscopiques d'agrégation-diffusion similaires à l'équation de Keller-Segel (17) et qui s'écrivent sous la forme

$$\frac{\partial \rho}{\partial t} = \mathsf{D}(\rho) + \lambda \operatorname{div}((\nabla K * \rho)\rho)\,,$$

où $\lambda \in \mathbb{R}_+^*$, D représente un opérateur de diffusion qui n'est pas forcément linéaire et K est un potentiel d'interaction attractif de la forme

$$\nabla K = \frac{x}{|x|^{\mathrm{a}}},$$

avec $\mathrm{a} > 0$. Comme indiqué en section 2.7, l'équation de Keller-Segel possède une propriété mathématiquement intéressante qui est celle d'explosion en temps fini lorsque la masse est au-dessus d'une certaine masse critique M_0^*, alors que l'équation est bien définie globalement en temps dans L^1 lorsque la masse est plus petite. De manière générale, on cherche à étudier la compétition entre l'agrégation, qui aura tendance à créer des explosions et à faire apparaître par exemple des masses de Dirac dans la solution, et la diffusion. On définit en fonction des paramètres de l'équation différentes zones caractérisées par l'effet dominant à petite échelle : régime d'**agrégation dominante**, régime de **diffusion dominante** et régime de **compétition équilibrée** (ou régime **critique**) lorsque les effets des deux termes sont du même ordre.

Une façon simple de repérer ces zones est de regarder l'effet de chaque opérateur face à un changement d'échelle. Comme on regarde des équations qui conservent la masse, on introduit la transformation $\mathsf{h}_n\rho := n^d\rho(nx)$ pour $n > 0$ qui fait un agrandissement vertical tout en conservant la masse. On obtient alors que l'opérateur d'agrégation $\mathcal{K}\rho := \operatorname{div}((\nabla K * \rho)\rho)$ vérifie

$$\mathsf{h}_n^{-1} \circ \mathcal{K} \circ \mathsf{h}_n = n^{\mathrm{a}}\, \mathcal{K}.$$

Cela indique que l'effet de l'agrégation augmente comme n^{a} lorsqu'on agrandit les variations verticales au détriment des variations horizontales, ce qui est finalement l'intensité de l'effet de $\mathcal{K}$ sur des variations locales. Si on a parallèlement que la diffusion vérifie pour n assez grand

$$\mathsf{h}_n^{-1} \circ \mathsf{D} \circ \mathsf{h}_n = n^{\alpha}\, \mathsf{D}, \tag{33}$$

alors l'effet de la diffusion sera plus fort localement lorsque $\alpha > \mathrm{a}$, et on s'attend à être dans le régime à diffusion dominante, avec des solutions régulières. Inversement, l'effet de la diffusion sera moins fort lorsque $\mathrm{a} > \alpha$, et on s'attend dans ce cas à être dans le régime à agrégation dominante. Remarquons que si ces relations sont aussi vérifiées pour n petit, alors l'agrégation sera plus forte sur les grandes échelles horizontales lorsqu'on est dans le cas de la diffusion dominante. On s'attend donc à avoir des solutions régulières mais plus confinées lorsque $\alpha > \mathrm{a}$ et des solutions moins régulières localement mais plus répandues pour les grandes valeurs de x sous l'action de la diffusion lorsque $\mathrm{a} > \alpha$.

Équation de Keller-Segel fractionnaire

On étudie en premier lieu dans le chapitre 6 le cas où $\mathsf{D} = \Delta^{\frac{\alpha}{2}}$ pour $\alpha \in]0, 2[$, ce qui est une généralisation du modèle classique de Keller-Segel (17) qui correspond au cas $\alpha = 2$ et $\mathrm{a} = d$. Notons qu'une telle généralisation est intéressante pour le cas de bactéries qui se déplacent selon un modèle de "run and tumble" telles que la bactérie *E. coli* [27] dont la modélisation mathématique introduite par D. Stoock [202] puis W. Alt [3] s'exprime sous la forme d'une équation cinétique dont le noyau de collision est non-local et du type opérateur de Lévy.

Dans ce cas, le changement d'échelle (33) a exactement lieu et on obtient le type de résultat attendu. Dans la partie à diffusion dominante en effet il y a existence en temps long dans des espaces de Lebesgue plus régulier que la condition initiale qui peut n'être que L^1. Dans la partie à agrégation dominante, on obtient une explosion en temps fini lorsque la masse est suffisamment grande et concentrée, et la convergence vers 0 lorsqu'une certaine norme de Lebesgue est suffisamment petite.

Une technique classique (voir par exemple [40]) pour montrer l'explosion en temps fini dans ce type d'équations consiste à dériver une inégalité différentielle pour les moments d'ordre 2 et de montrer que ceux-ci tendent vers 0 en temps fini, ce qui indique une contradiction à partir d'un certain moment. Dans le cas $\alpha = \mathrm{a} = 2$, cela s'écrit simplement puisque si $\rho = \rho(t, x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^1)$ est une solution de (17) alors en intégrant par partie et en utilisant la symétrie

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d} |x|^2 \rho(\mathrm{d}x)\right) &= \int_{\mathbb{R}^d} \Delta(|x|^2)\rho(\mathrm{d}x) - \lambda \iint_{\mathbb{R}^{2d}} \nabla K(x-y)\cdot(x-y)\,\rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &= 2dM_0 - \lambda M_0^2,\end{aligned}$$

d'où l'on déduit que si $\lambda M_0 \geq 2d$, il y aura une explosion en temps fini.

Dans le cas du Laplacien fractionnaire et de $\mathrm{a} \neq 2$, l'idée est de remplacer $|x|^2$ par un autre poids m adapté aux paramètres (a, α). Or le Laplacien fractionnaire d'une fonction qui se comporte pour les grands $|x|$ comme $|x|^k$ n'est défini que si $k < \alpha$, donc on prend plutôt $|x|^k$ pour $|x| > 1$. De plus, au voisinage de $x = 0$, on veut que la singularité ne soit pas trop forte pour que le Laplacien fractionnaire de m reste borné et en même temps que le terme provenant de l'agrégation, qui s'écrit alors

$$\lambda \iint_{\mathbb{R}^{2d}} \nabla K(x-y)\cdot(\nabla m(x) - \nabla m(y))\,\rho(\mathrm{d}x)\rho(\mathrm{d}y),$$

soit borné inférieurement par quelque chose qui ne tend pas vers 0 lorsque $x - y \to 0$. On arrive donc à un poids qui s'écrit

$$m(x) = |x|^{\mathrm{a}}\,\mathbb{1}_{|x|<1} + |x|^k\,\mathbb{1}_{|x|>1}.$$

On doit en fait également remplacer les indicatrices par des fonctions plus régulières puisque l'on veut que $\Delta^{\frac{\alpha}{2}} m$ soit borné même si $\alpha > 1$. Le résultat d'explosion en temps fini s'écrit finalement sous la forme suivante.

Théorème 8. *Soient* $(\alpha, \mathrm{a}) \in [0,2) \times [1,d)$ *tels que* $\alpha < \mathrm{a}$, $k \in]0, \alpha[$ *et* $\rho \in C^0(\mathbb{R}_+, L^1(m))$ *une solution positive et paire de l'équation de Keller-Segel* (17) *de condition initiale* $\rho^{\mathrm{in}} \in L^1(m)$ *vérifiant*

$$\int_{\mathbb{R}^d} \rho^{\mathrm{in}}(x)\, \langle x \rangle^k \,\mathrm{d}x \leq C^* \lambda^{\frac{k}{2(\mathrm{a}-k)}} M_0^{\frac{2\,\mathrm{a}-k}{2(\mathrm{a}-k)}} \qquad \text{si } \alpha > 1$$
$$\int_{\mathbb{R}^d} \rho^{\mathrm{in}}(x) |x|^k \,\mathrm{d}x \leq C_2^* M_0 \ \text{ et } \lambda M_0 \geq C_3^* \qquad \text{si } \alpha < 1,$$

pour certaines constantes C^*, C_2^*, C_3^* *dépendant uniquement de* d, a, α *et* k*. Alors la solution cesse d'être bien définie en temps fini.*

D'un autre côté, en regardant le comportement de normes de Lebesgue à poids, on obtient comme attendu le caractère bien posé et un gain d'intégrabilité locale lorsque $\mathrm{a} < \alpha$, puisque pour tout $p \in]1, p_{\mathrm{a}}[$ on a

$$\|\rho\|_{L^p} \leq C M_0\, t^{-\frac{d}{\alpha p'}} + C_\lambda(M_0).$$

où $p_{\mathrm{a}} = \frac{d}{d - \mathrm{a}}$. Lorsque $\alpha < \mathrm{a}$, en posant $p_{\mathrm{a},\alpha} := \frac{d}{d + \alpha - \mathrm{a}}$, alors on observe tout de même la diffusion due au Laplacien fractionnaire si la solution est suffisamment étalée initialement dans le sens où pour $p \in (p_{\mathrm{a},\alpha}, p_{\mathrm{a}})$, $\|\rho^{\mathrm{in}}\|_{L^p} < C_{\lambda,p}(M_0)$ une certaine constante positive. Dans ce cas, il existe $C = C_{\mathrm{a},\alpha,p}(\rho^{\mathrm{in}}) > 0$ tel que

$$\|\rho\|_{L^p} \leq C M_0\, t^{-\frac{d}{\alpha p'}}. \tag{34}$$

Dans le cas critique $\mathrm{a} = \alpha$, on retrouve un peu des deux types de comportements, puisqu'on a toujours un étalement mais aussi un gain d'intégrabilité locale et la condition d'étalement se transforme en condition de petite masse initiale. Plus précisément (34) a lieu si $\lambda M_0 < C_{d,\mathrm{a},p}$ pour une constante explicite dépendant uniquement de d, a et p. Les résultats sont résumés en Figure 3.

Dans le cas où les bonnes normes de Lebesgue restent finies, on peut prouver l'existence et l'unicité des solutions en montrant un résultat de stabilité avec les distances de Wasserstein, similaire à celui déjà utilisé en chapitre 1 pour quantifier la limite semi-classique. Une remarque est à faire cependant sur l'avantage ici d'utiliser des distances de Wasserstein. L'équation de Keller-Segel fractionnaire se réécrit facilement du point de vue Lagrangien en utilisant la théorie des probabilités. En effet, elle correspond à l'équation régissant la dynamique de la loi du processus vérifiant l'équation stochastique suivante

$$\mathrm{d}X = E(t, X)\,\mathrm{d}t + \mathrm{d}B_\alpha,$$

où $E(t,x) = -\lambda\, (\nabla K * \rho)(t,x)$ et $B_\alpha = B_\alpha(t)$ désigne un processus de Lévy α-stable. Dans ce formalisme, les distances de Wasserstein se réécrivent simplement en termes d'espérance sous la forme

$$W_p(\rho, \tilde\rho) = \left(\inf_{\gamma \in \Pi(\rho, \tilde\rho)} \{ \mathbb{E}(|X - Y|^p), (X,Y) \sim \gamma \} \right)^{\frac{1}{p}}, \tag{35}$$

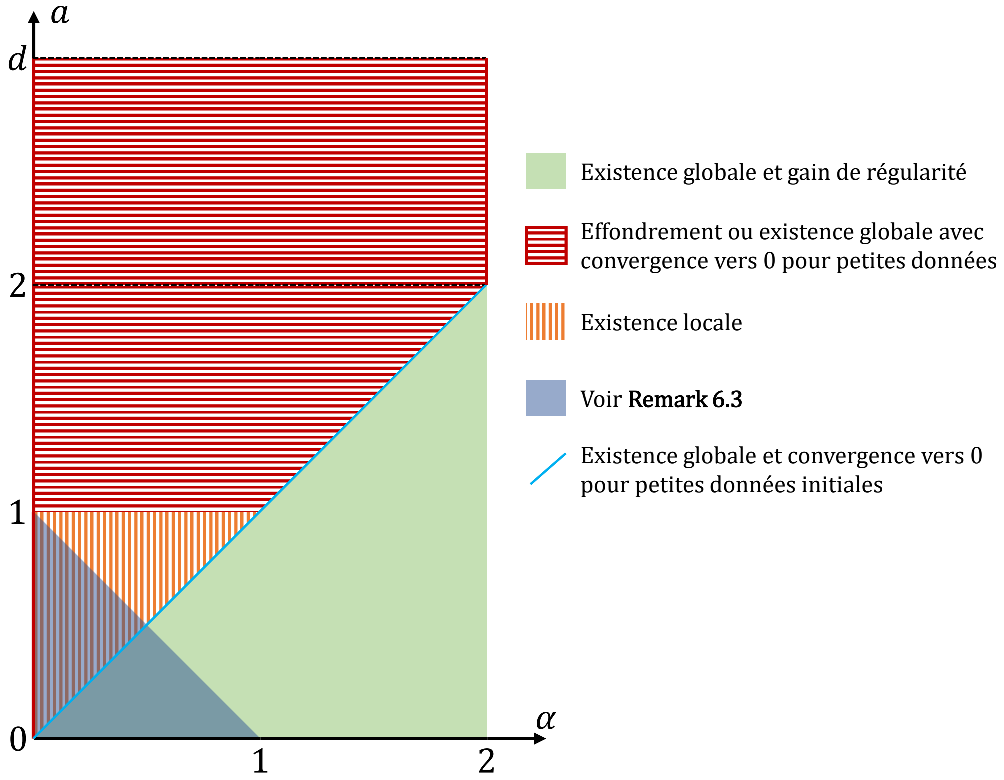

FIGURE 3 : Résultats du chapitre 6. On précise en particulier que pour $d > 2$, les résultats sont aussi valides pour le segment $(\alpha, \mathrm{a}) \in \{2\}\times]0, d[$. Comme indiqué en remarque 6.3, les résultats ne sont plus valides pour $\alpha + \mathrm{a} < 1$ uniquement à cause du fait que le potentiel devient trop grand à l'infini. Cependant, comme on s'intéresse à la singularité de l'interaction, les même résultats restent valides par exemple pour des noyaux de la forme $\nabla K = \frac{x}{|x|^{\mathrm{a}}}\mathbb{1}_{|x|<1} + \frac{x}{|x|^b}\mathbb{1}_{|x|\geq 1}$ pour $b > 1 - \alpha$.

où $\Pi(\rho, \tilde{\rho})$ désigne l'ensemble des couplages entre ρ et $\tilde{\rho}$. L'Appendice A fournit quelques détails supplémentaires sur les distances de Wasserstein. Or on peut en particulier prendre deux processus X et Y vérifiant cette équation stochastique avec le même processus de Lévy, et on a alors

$$\mathrm{d}(X - Y) = (E(X) - E(Y))\,\mathrm{d}t$$

ce qui fait que la démonstration de stabilité peut se faire de façon identique au cas déterministe où il n'y a pas de processus de Lévy.

Équation de Keller-Segel avec un p-Laplacien

Enfin, le chapitre 7 traite le cas où $\mathsf{D} = \Delta_p$ est le p-Laplacien. Celui-ci est défini par

$$\Delta_p\rho = \operatorname{div}\Big(|\nabla\rho|^{p-2}\nabla\rho\Big)$$

et donc le cas du Laplacien classique correspond cette fois au cas $p = 2$. Ce type de diffusion n'est finalement rien d'autre que celle où le coefficient de diffusion est proportionnel à une certaine puissance du gradient et apparaît dans les mouvements des tas de sable (voir par exemple [10, 91]).

On peut dans ce cas faire le même raisonnement de changement d'échelle et on obtient alors

$$\Delta_p(\mathsf{h}_n\rho) = \operatorname{div}\Big(|\nabla(\mathsf{h}_n\rho)|^{p-2}\nabla(\mathsf{h}_n\rho)\Big) = n^{p(d+1)-2d}\,\mathsf{h}_n \operatorname{div}\Big(|\nabla\rho|^{p-2}\nabla\rho\Big)\,,$$

ce qui fait qu'en posant $\alpha_p := p\,(d+1) - 2d$ on obtient exactement (33) pour $\alpha = \alpha_p$. Le chapitre montre qu'à nouveau, pour $\mathrm{a} < \alpha_p$, on obtient des estimées uniformes en temps, ainsi que si $\mathrm{a} = \alpha_p$ et que la masse est initialement suffisamment petite. Les résultats sont résumés en Figure 4.

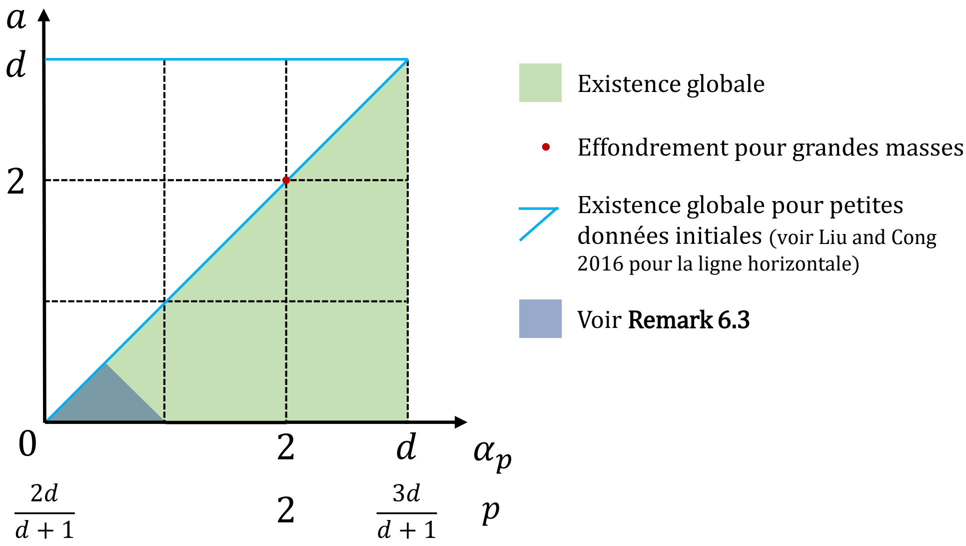

Figure 4 : Résumé des résultats du Chapitre 7. Le cas $\mathrm{a} + \alpha_p < 1$ doit être priori exclu, puisque des moments d'ordre au moins $1 - \mathrm{a}$ sur ρ sont requis pour donner du sens au terme $(\nabla K * \rho)$, et Δ_p propage des moments d'ordre au plus α_p (voir Lemme 7.3). On renvoie à nouveau à la remarque 6.3.

On s'attend bien sûr à avoir aussi un effondrement en temps fini si $\alpha < \mathrm{a}$ avec a assez grand au moins dans le cas de données initiales radiales, cependant la forme très singulière du terme non-linéaire empêche d'utiliser une technique similaire au chapitre précédent, et de nouvelles stratégies doivent donc être trouvées pour ce type de problème.

5 Perspectives

De nombreuses questions apparaissent naturellement à l'aboutissement de cette thèse. On en liste ici quelques-unes qui semblent être le plus directement en lien avec les résultats obtenus.

De la mécanique quantique aux modèles cinétiques

Une première question liée à la Partie I est celle de la limite de champ moyen quantique de l'équation de Schrödinger à N corps (4) vers l'équation de (Hartree). La recherche concernant cette question est en effet assez active ces dernières années, parallèlement à celle de la dérivation de l'équation de (Vlasov) depuis un système de N particules. Si la limite de champ moyen quantique est bien comprise pour $\hbar$ fixé, il manque encore des résultats concernant la limite uniformément en $\hbar$ pour ce cas. Un problème très proche est la quantification en fonction de N et $\hbar$ de la limite de l'équation de Schrödinger (4) directement vers l'équation de (Vlasov). Cependant, des avancées ont été faites pour le cas de fermions [183, 186] où un tel résultat est démontré mais sous des conditions de régularité pour les solutions de l'équation de (Hartree) qui restent un problème ouvert. Les résultats de propagation de normes de Lebesgue semi-classiques semblent donc être un bon premier pas vers des résultats de type champ-moyen.

Une autre piste intéressante est d'essayer d'obtenir des estimées dans des normes plus fortes. Des résultats récents semblent assez prometteurs dans cette direction [188, 187]. Ceux-ci reposent sur des inégalités de type fort-faible qui utilisent principalement la régularité de la donnée limite. La connaissance de la régularité semi-classique des solutions de l'équation de (Hartree) pourrait encore une fois apporter de nouvelles pistes.

Comportement des opérateurs non-locaux

Concernant la Partie II, il serait intéressant de réussir à démontrer des résultats de positivité semblables à ceux démontrés au Chapitre 3 dans le cas non-homogène en espace correspondant aux deux chapitres suivants. En effet, les deux chapitres 4 et 5 utilisent fortement le fait que l'on connaisse les états stationnaires pour démontrer la coercivité microscopique à poids. Des résultats de positivité permettraient alors d'obtenir des résultats d'hypocoercivité pour des équations dont les solutions sont peu régulières. Citons en particulier [51] dans lequel les auteurs utilisent ce type de technique dans le cas d'un opérateur de type Boltzmann linéaire et de solutions non homogènes en espace.

Une question ouverte dans le cas du Chapitre 3 est d'obtenir des estimées plus précises de régularité et des bornes uniformes sur les solutions et l'état stationnaire de l'équation de Fokker-Planck fractionnaire (FFP). En effet, il a été montré dans [197] que des discontinuités peuvent apparaître lorsque le champ E n'est pas assez régulier. Dans le cas d'un champ fortement confinant, il ne paraît cependant même pas évident d'obtenir une borne L^∞ localement en espace. Cependant, on sait montrer dans l'autre direction que si l'état d'équilibre est borné et à décroissance polynomiale alors on peut avoir des champs de force fortement confinants (voir la Proposition C.28 de l'Appendice C).

De façon plus générale, il serait intéressant d'étudier l'existence et l'unicité de solutions dans un sens plus faible lorsque le champ E est très irrégulier et que même les estimées dans des espaces de Lebesgue ne sont plus possibles. On peut alors s'inspirer de la théorie des solutions renormalisées introduite dans [77, 78].

Pour le Chapitre 5, une extension semble envisageable dans les cas où il n'y a plus de coercivité microscopique à poids en utilisant des inégalités de Poincaré plus faibles et leurs équivalents pour les différents opérateurs considérés. Le taux semble alors fortement dépendre de l'espace de départ considéré et est dû à la lente mise à l'équilibre des vitesses.

Équations d'agrégation-diffusion

Dans la partie III, la présence ou non d'un effondrement en temps fini dans le cas à agrégation dominante mais où la force devient moins singulière reste un problème ouvert intéressant. En effet, en l'absence d'effondrement, cela suggèrerait que ce serait alors l'effet des forces d'attraction à distance qui apporterait une certaine régularité à la solution.

Remarquons aussi que la méthode utilisée semble pouvoir se généraliser à d'autres équations de type agrégation-diffusion tel que le cas de la diffusion non-linéaire rapide.

Première partie

From Quantum Mechanics to Kinetic Models

Chapitre 1

Propagation of Moments and Semiclassical Limit from Hartree to Vlasov Equation

Abstract

In this chapter, we prove a quantitative version of the semiclassical limit from the Hartree to the Vlasov equation with singular interaction, including the Coulomb potential. To reach this objective, we also prove the propagation of velocity moments and weighted Schatten norms which implies the boundedness of the space density of particles uniformly in the Planck constant.

Résumé

Dans ce chapitre, on montre une version quantitative de la limite semi-classique de l'équation de Hartree vers l'équation de Vlasov avec des potentiels singuliers, en incluant le potentiel Coulombien. Pour atteindre cet objectif, on démontre aussi que les moments en vitesse et des normes de Schatten à poids sont propagées en temps, ce qui implique que la densité spatiale reste bornée uniformément en espace et en la constante de Planck.

Contents

1.1 Introduction

1.1.1 Presentation of the problem

In this chapter, we consider the non-relativistic quantum and classical equations which describe the evolution of a density of infinitely many particles in the kinetic mean field regime, called respectively the Vlasov and the Hartree equation. The interaction between particles is described by a mean field potential $V = V(x)$ depending only on the space variable $x \in \mathbb{R}^d$ with $d \geq 2$ and which is defined by

$$V := K * \rho = \int_{\mathbb{R}^d} K(x-y)\rho(y)\,\mathrm{d}y,$$

where ρ is the spatial density and K is an even kernel describing the interaction between two particles. The force field can then be written

$$E := -\nabla V.$$

Typically, we have in mind the pair interaction potential $K(x) = \frac{\pm 1}{|x|^a}$ with $a \in [-2, d-1)$. The most physically relevant case is the case of the Coulomb interaction $a = d-2$ for

$d \geq 3$ or $K(x) = \pm\ln(|x|)$ in the two dimensional case. It can describe the interaction of charged particles as well as a system of point masses in gravitational interaction, the force being repulsive when K is positive and attractive in the converse case.

In the classical case, the kinetic density of particles $f = f(t,x,\xi)$ is a non-negative function of time $t \in \mathbb{R}_+$, space and momentum $\xi \in \mathbb{R}^d$ and the space density is given by

$$\rho(x) := \int_{\mathbb{R}^d} f(t,x,\xi)\,\mathrm{d}\xi.$$

The evolution of the kinetic density is then given by the well-known Vlasov equation

$$\partial_t f + \xi\cdot\nabla_x f + E\cdot\nabla_\xi f = 0. \tag{Vlasov}$$

Remark also that by defining the Hamiltonian

$$H := \frac{|\xi|^2}{2} + V,$$

we can write the (Vlasov) equation as

$$\partial_t f = \{H, f\},$$

where $\{\cdot,\cdot\}$ is the Poisson bracket defined by

$$\{f,g\} = \nabla_x f\cdot\nabla_\xi g - \nabla_\xi f\cdot\nabla_x g.$$

On the other hand, in the formalism of quantum mechanics, a particle is described by a wave function $\psi \in L^2 = L^2(\mathbb{R}^d,\mathbb{C})$ verifying $\|\psi\|_{L^2} = 1$. Under the action of the potential V, its evolution is governed by the following Schrödinger equation

$$i\hbar\,\partial_t\psi = -\frac{\hbar^2}{2}\Delta\psi + V\psi, \tag{1.1}$$

where $\hbar = \frac{h}{2\pi}$ is the reduced Planck constant. In the more general case of systems with mixed states, the density of particles is described by a trace class and self-adjoint density operator, $\boldsymbol{\rho}$, which by the Spectral theorem can be seen as a superposition of pure orthonormal states $(\psi_j)_{j\in J} \in (L^2)^J$ for a given $J \subset \mathbb{N}$ by writing

$$\boldsymbol{\rho}\,\varphi := \int_{\mathbb{R}^d} \varrho(x,y)\,\varphi(y)\,\mathrm{d}y = \sum_{j\in J}\lambda_j|\psi_j\rangle\langle\psi_j|\varphi. \tag{1.2}$$

This is a Hilbert-Schmidt operator of kernel

$$\varrho(x,y) = \sum_{j\in J}\lambda_j\overline{\psi_j(y)}\psi_j(x).$$

Given a density operator, the spatial density is defined as the diagonal of the kernel

$$\rho(x) := \varrho(x,x) = \sum_{j\in J}\lambda_j|\psi_j|^2,$$

and the Hamiltonian is the following operator

$$H = -\frac{\hbar^2}{2}\Delta + V,$$

where $V = K * \rho$ is identified with the operator of multiplication by $V(x)$. We can then rewrite (1.1) for each ψ_j as $\partial_t \psi_j = \frac{1}{i\hbar} H\psi_j$ and we deduce that the density operator verifies the so called Hartree equation

$$\partial_t \boldsymbol{\rho} = \frac{1}{i\hbar}\left[H, \boldsymbol{\rho}\right], \tag{Hartree}$$

where $[\cdot,\cdot]$ is the Lie bracket defined by

$$[A, B] = AB - BA.$$

The main goal of the present chapter is to obtain a *quantitative estimate* of the semi-classical limit from the (Hartree) to the (Vlasov) equation, which means the limit when $h = 2\pi\hbar \to 0$. This limit was first investigated in a non-quantitative way using compactness methods by Lions and Paul [152], Markowich and Mauser [158] and then by Gerard et al [110], Gasser et al [97], Ambrosio et al [5, 4], Graffi et al [109]. On the other hand, Athanassoulis et al [11] prove quantitative estimates in L^2 norm in the case of sufficiently smooth potentials and Amour et al [6, 7] show that the rate can be improved in the case of very smooth potentials. More recently, some improvements on the requirement of regularity of the potential K have been done in Benedikter et al [25] by considering trace and Hilbert-Schmidt norms and a mixed semiclassical and mean-field limit, and by Golse et al [104] and Golse and Paul [105] using quantum pseudo-distances created on the model of the Wasserstein-Monge-Kantorovitch distances. This strategy allows them to prove estimates that do not require any assumption of regularity on the initial data. However, all these works still require at least Lipschitz regularity of the potential, which does not include singular interactions like the Coulomb potential.

Recent attempts on generalizing these results to more singular potentials in the case of fermionic systems can be found in the works by Porta et al [183] and Saffirio [186], where a joint mean-field and semiclassical limit is obtained. However, it requires regularity assumptions on the solution of the Hartree equation whose propagation is still an open problem. The closely related problem of the mean field limit from the N-body Schrödinger to the Hartree equation has been also investigated a lot. Weak convergence results have been first obtained in [20, 89, 19] for the Coulomb potential. See also [212] for the one dimensional case. Quantitative results have been established in [185, 182, 104, 171, 105, 107, 108] for Bosons and in [96, 95, 26, 24, 13, 181, 183, 180] for Fermions. Remark that some of these works use a joint mean-field and semiclassical limit, however they always require at least a Lipschitz potential or an assumption of regularity on the solution of the Hartree equation.

An other possible way to derive the Vlasov equation is the classical mean-field limit. It is also a closely related problem. Results for non-smooth potentials can be found for example in [116, 117, 139, 127, 140]. The major obstacle here is the absence of regularity in the N-body problem, which is the reason why all results with unbounded pair interaction potentials need a cut-off on the force field.

Other results about the mean-field limit are the convergence of the minimizers of the N-particles energy towards the mean-field energy. We refer for example to [92] and references therein.

In order to get semiclassical estimates for more general pair potentials K, our strategy consists in requiring more regularity on the initial data and proving that it implies regularity at the level of the mean-field potential V. The propagation of moments is inspired from [153] and [152]. The semiclassical limit is mostly an adaptation of [105] and of the proof of uniqueness for the Vlasov equation given in [157]. Some interesting improvements for the uniqueness can be found in [163, 121].

Finally, notice that the global well-posedness in Sobolev and Schatten spaces and conservation of the energy have been treated in [100, 101, 118, 49, 125, 62, 142]. In particular, our hypotheses on finite quantum moments of order n require the equation to be well-posed in the corresponding H^n Sobolev space. However, as we will see, even if quantum moments can be interpreted as H^n norms, the above mentioned papers do not prove the propagation of these norms uniformly with respect to $\hbar$, which is one the main results of this chapter.

1.1.2 Notations and tools

We describe in this section the main notations that we will use. Since we are in the semiclassical regime, most of our results have to be true in the limit and are inspired from the classical results. Therefore, our notations try to be close for the classical objects and their quantum counterpart. When comparing quantum and classical objects, we will sometimes add $\hbar$ in the notation to denote the quantum objects.

Functional spaces

Since most of the functional spaces we use will be defined for functions defined on $\mathbb{R}^d$, we will often write $X = X(\mathbb{R}^d, \mathbb{C})$, as for example in the case of the Lebesgue spaces $L^p := L^p(\mathbb{R}^d, \mathbb{C})$. When working on the phase space $\{(x, \xi) \in \mathbb{R}^{2d}\}$ we will write $L^p_{x,\xi} := L^p(\mathbb{R}^{2d}, \mathbb{C})$. Some other standard functional spaces we will use are the weak and weighted Lebesgue spaces, defined reciprocally by

$$\begin{aligned} L^{p,\infty} &:= \{f \text{ measurable}, \forall \lambda > 0, |\{|f| > \lambda\}| \leq C/\lambda^p\} \\ L^{\infty,\infty} &:= L^\infty \\ L^p(m) &:= \{f \text{ measurable}, fm \in L^p\}. \end{aligned}$$

Moreover, we will denote by $\mathcal{P}(X)$ the space of probability measures on some space X. We will need the equivalent of some of these spaces in the quantum picture. The quantum equivalent of the integral on the phase space is the trace which for an operator $\boldsymbol{\rho}$ in the form (1.2) can be written

$$\mathrm{Tr}(\boldsymbol{\rho}) = \int_{\mathbb{R}^d} \varrho(x,x)\,\mathrm{d}x = \sum_{j\in J} \lambda_j.$$

The trace is defined more generally for trace class operators. We refer to [199] for the general definition and additional properties. In order to define the equivalent of Lebesgue norms, let us first recall the definition of the Schatten norm of a trace class operator A for $p \in [1,+\infty)$

$$\|A\|_p := \mathrm{Tr}(|A|^p)^{1/p}$$
$$\|A\|_\infty := \|A\|_{\mathscr{B}},$$

where $\mathscr{B} = \mathscr{B}(L^2)$ is the space of bounded operator on L^2 and $|A| = \sqrt{A^*A}$. We will more precisely use a rescaled version of these norms defined for $r \in [1,+\infty]$ by

$$\|\boldsymbol{\rho}\|_{\mathcal{L}^r} := h^{-d/r'}\|\boldsymbol{\rho}\|_r, \tag{1.3}$$

where $r' = \frac{r}{r-1}$ denotes the Hölder conjugate of r. They play the role of the $L^r_{x,\xi}$ norm for the quantum density operators. The space of quantum probability measures corresponds to the space of normalized hermitian operators defined by

$$\mathscr{P} := \{\boldsymbol{\rho} \in \mathscr{B}(L^2), \boldsymbol{\rho} = \boldsymbol{\rho}^* \geq 0, \mathrm{Tr}(\boldsymbol{\rho}) = 1\}.$$

Remark that since $\boldsymbol{\rho}$ will usually be a nice compact operator, for general unbounded operators $A \in \mathcal{L}(L^2)$, we can define $\mathrm{Tr}(A\boldsymbol{\rho}) := \mathrm{Tr}\Big(\boldsymbol{\rho}^{1/2}A\boldsymbol{\rho}^{1/2}\Big)$ even if $A\boldsymbol{\rho}$ is not a bounded operator.

Momentum

We recall that the quantum equivalent of the classical momentum ξ is the following unbounded operator from L^2 to $(L^2)^d$

$$\mathsf{p} := -i\hbar\nabla.$$

Its formal adjoint for the scalar product defined by $\langle u, w\rangle_{(L^2)^d} = \int_{\mathbb{R}^d} \overline{u}\cdot w$ is then defined by

$$\mathsf{p}^* = -i\hbar\,\mathrm{div} = \mathsf{p}\,\cdot,$$

which leads to the following notations

$$-\hbar^2\Delta = |\mathsf{p}|^2$$
$$H = \frac{|\mathsf{p}|^2}{2} + V.$$

Wigner Transform

There exists several ways to try to associate a density over the phase space to a density operator, one of them being the Wigner transform and its non-negative but smoothed version called Husimi transform defined reciprocally for $h = 1$ by

$$\begin{aligned} w(\boldsymbol{\rho})(x,\xi) &:= \int_{\mathbb{R}^d} e^{-2i\pi y\cdot\xi}\varrho\left(x+\frac{y}{2}, x-\frac{y}{2}\right)\mathrm{d}y = \mathcal{F}(\tilde{\varrho}_x)(\xi) \\ \widetilde{w}(\boldsymbol{\rho}) &:= w(\boldsymbol{\rho}) * G, \end{aligned}$$

where $\tilde{\varrho}_x(y) = \varrho(x+y/2, x-y/2)$ and $G(z) = \frac{1}{\pi^d}e^{-|z|^2}$ with $z := (x,\xi)$ and we used the following convention for the Fourier transform

$$\mathcal{F}(u)(\xi) := \int_{\mathbb{R}^d} e^{-2i\pi x\cdot\xi}u(x)\,\mathrm{d}x.$$

We refer for example to [152] and [104] for more details and mathematical results. Given $\boldsymbol{\rho}$ solution of the (Hartree) equation we will write its Wigner and Husimi transforms respectively

$$\begin{aligned} f_\hbar(x,\xi) = w_\hbar(\boldsymbol{\rho})(x,\xi) &:= \frac{1}{h^d}w(\boldsymbol{\rho})\left(x,\frac{\xi}{h}\right) \\ \tilde{f}_\hbar(x,\xi) &:= f_\hbar * G_\hbar, \end{aligned}$$

where $G_\hbar(z) = \frac{1}{(\pi\hbar)^d}e^{-|z|^2/\hbar} = g_\hbar(x)g_\hbar(y)$ with $g_\hbar(x) = \frac{1}{(\pi\hbar)^{d/2}}e^{-|x|^2/\hbar}$. We also define the quantum velocity moments by

$$M_n := \mathrm{Tr}(|\mathsf{p}|^n\boldsymbol{\rho}) = \int_{\mathbb{R}^{2d}} f_\hbar\,|\xi|^n\,\mathrm{d}x\,\mathrm{d}\xi.$$

Remark that the scaling of the quantum Lebesgue norm $\mathcal{L}^p$ can be understood by looking at the Wigner transform and noticing that when $r = 1$ or $r = 2$

$$\begin{aligned} \iint_{\mathbb{R}^{2d}} f_\hbar\,\mathrm{d}x\,\mathrm{d}\xi &= \|\boldsymbol{\rho}_\hbar\|_{\mathcal{L}^1} \\ \|f_\hbar\|_{L^2_{x,\xi}} &= \|\boldsymbol{\rho}_\hbar\|_{\mathcal{L}^2}. \end{aligned}$$

Moreover, when $r > 2$ and $\boldsymbol{\rho}_\hbar$ is a superposition of coherent states, then

$$\begin{aligned} \|f_\hbar\|_{L^r_{x,\xi}} &\leq \|\boldsymbol{\rho}_\hbar\|_{\mathcal{L}^r} \\ \|f_\hbar\|_{L^r_{x,\xi}} &\underset{h\to 0}{\longrightarrow} \|\boldsymbol{\rho}_\hbar\|_{\mathcal{L}^r}. \end{aligned}$$

See Section 1.7 for the proof and other results for coherent states.

Semiclassical Wasserstein pseudo-distances.

A last useful tool in the study of uniqueness and stability estimates for the Vlasov equation is the Wasserstein-(Monge-Kantorovich) distance W_p which can be defined for any

$p \in [1, \infty]$. We refer for example to the books by Villani [209] and Santambrogio [191]. As introduced in [105], we will use a quantum equivalent of the W_2 distance. We first introduce the notion of coupling between a density operator and a classical kinetic density. Let $\boldsymbol{\gamma} \in L^1(\mathbb{R}^{2d}, \mathscr{P})$. We say that $\boldsymbol{\gamma}$ is a semiclassical coupling of $f \in L^1 \cap \mathcal{P}(\mathbb{R}^{2d})$ and $\boldsymbol{\rho} \in \mathscr{P}$ and we write $\boldsymbol{\gamma} \in \mathcal{C}(f, \boldsymbol{\rho})$ when

$$\mathrm{Tr}(\boldsymbol{\gamma}(z)) = f(z)$$
$$\int_{\mathbb{R}^{2d}} \boldsymbol{\gamma}(z)\,\mathrm{d}z = \boldsymbol{\rho}.$$

Then we define the semiclassical Wasserstein-(Monge-Kantorovich) pseudo-distance in the following way

$$W_{2,\hbar}(f, \boldsymbol{\rho}) := \left(\inf_{\gamma \in \mathcal{C}(f,\boldsymbol{\rho})} \int_{\mathbb{R}^{2d}} \mathrm{Tr}(\mathbf{c}_\hbar(z)\boldsymbol{\gamma}(z))\,\mathrm{d}z \right)^{\frac{1}{2}}, \tag{1.4}$$

where $\mathbf{c}_\hbar(z)\varphi(y) = (|x-y|^2 + |\xi - \mathsf{p}|^2)\,\varphi(y)$, $z = (x, \xi)$ and $\mathsf{p} = -i\hbar\nabla_y$. This is not a distance but it is comparable to the classical Wasserstein distance W_2 between the Wigner transform of the quantum density operator and the normal kinetic density, in the sense of the following Theorem

Theorem 1.1 (Golse & Paul [105])**.** *Let $\boldsymbol{\rho} \in \mathscr{P}$ and $f \in \mathcal{P}(\mathbb{R}^{2d})$ be such that*

$$\int_{\mathbb{R}^{2d}} f(x,\xi) \left(|x|^2 + |\xi|^2\right) \mathrm{d}x\,\mathrm{d}\xi < \infty.$$

Then one has $W_{2,\hbar}(f, \boldsymbol{\rho})^2 \geq d\hbar$ and for the Husimi transform $\tilde{f}_\hbar$ of $\boldsymbol{\rho}$, it holds

$$W_2(f, \tilde{f}_\hbar)^2 \leq W_{2,\hbar}(f, \boldsymbol{\rho})^2 + d\hbar. \tag{1.5}$$

See also [106] for more results about this pseudo-distance and Section 1.7 for the particular case of coherent states.

1.1.3 Main results

We will use this pseudo-distance to get explicit speed of convergence in $\hbar$ of the solution $\boldsymbol{\rho}_\hbar$ of (Hartree) equation to the solution f of (Vlasov) equation. For the classical density f, we consider conditions which ensure existence and uniqueness of the solution and the boundedness of ρ as claims the following theorem

Theorem 1.2 (Lions & Perthame [153], Loeper [157])**.** *Assume $f^{\mathrm{in}} \in L^\infty_{x,\xi}(\mathbb{R}^6)$ verify*

$$\int_{\mathbb{R}^6} f^{\mathrm{in}}\,|\xi|^{n_0}\,\mathrm{d}x\,\mathrm{d}\xi < C \textit{ for a given } n_0 > 6, \tag{1.6}$$

and for all $R > 0$,

$$\sup_{(y,w)\in\mathbb{R}^6} \mathrm{ess}\{f^{\mathrm{in}}(y + t\xi, w), |x-y| \leq Rt^2, |\xi - w| \leq Rt\} \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty_x L^1_\xi). \tag{1.7}$$

Then there exists a unique solution to the (Vlasov) equation with initial condition $f_{t=0} = f^{\mathrm{in}}$. Moreover, in this case, the spatial density verifies

$$\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty). \tag{1.8}$$

This is actually proved for the Vlasov-Poisson equation only (i.e. $K = \frac{1}{|x|}$) but the proof would works for less singular potentials verifying the assumptions of the following Theorem 1.3. Actually, the proof we make for the quantum case can be easily adapted to the classical case, which implies for example that this result holds in dimension 3 for

$$K = \frac{1}{|x|^a} \text{ for any } a \in (-1, 4/5), \tag{1.9}$$

and for all $t \in [0, T_{\max}]$ when $a \in [4/5, 8/7)$. The strategy to prove the above theorem is to obtain a Gronwall's inequality for moments. Our first Theorem uses the same strategy in the semiclassical picture to prove the propagation of quantum velocity moments.

Theorem 1.1. *Let $r \in [1, \infty]$, $\mathfrak{b}_n := \frac{nr'+d}{n+1}$ and $\nabla K \in L^{\mathfrak{b},\infty}$ for a given $\mathfrak{b} \in (\max(\mathfrak{b}_4, \mathfrak{b}_n), +\infty]$ and assume $\boldsymbol{\rho}_\hbar$ verify the (Hartree) equation for $t \in [0, T]$ with initial condition $\boldsymbol{\rho}_\hbar^{\text{in}} \in \mathscr{P} \cap \mathcal{L}^r$ such that M_n^{in} is bounded independently of $\hbar$ for a given $n \in 2\mathbb{N}$. Then there exists $T > 0$ and $\Phi \in C^0[0, T)$ such that for any $t \in [0, T)$*

$$M_n \leq \Phi(t). \tag{1.10}$$

Moreover, $T = +\infty$ when $\mathfrak{b} \geq \mathfrak{b}_2 = \frac{2r'+d}{3}$. In particular, if $K = \frac{1}{|x|^a} \in L^{d/a,\infty}$ and $r = \infty$, then we require

- *$a \in \left(-1, \frac{2}{3}\right)$ if $d = 2$,*
- *$a \in \left(-1, \frac{8}{7}\right)$ if $d = 3$,*

and $T = +\infty$ when

- *$a \in \left(-1, \frac{1}{2}\right)$ if $d = 2$,*
- *$a \in \left(-1, \frac{4}{5}\right)$ if $d = 3$.*

From the quantum kinetic interpolation inequalities (1.21), we obtain the following corollary.

Corollary 1.2. *Under the assumptions of Theorem 1.1,*

$$\|\rho\|_{L^p},$$

is bounded on $[0, T)$ independently of $\hbar$ for any $p \in [1, p_n]$, where $p_n' = r' + \frac{d}{n}$.

Remark 1.3. *As it can be seen in the proof, when $\mathfrak{b} \geq \max(\mathfrak{b}_n, \mathfrak{b}_N)$ for $4 \leq n < N \in 2\mathbb{N}$ and there is propagation of moments of order n, then the propagation of all higher moments M_N holds with $T = +\infty$. In particular, in dimension $d = 3$, if $r = \infty$, as long as the moments M_4 are finite, then all the higher moments are propagated for $\mathfrak{b} \geq \frac{7}{5}$, or equivalently if $K = |x|^{-a}$, for $a \leq \frac{8}{7}$, which includes the Coulomb case.*

Remark 1.4. *As explained in Section 1.3.2, the constraint $a > -1$ could be easily removed by assuming bounded space moments $N_k = \operatorname{Tr}\left(|x|^k \boldsymbol{\rho}\right)$ for a given $k \leq n$, allowing for polynomial growth of K for large $|x|$.*

The next theorem is about the following semiclassical convergence result which uses only hypothesis on initial velocity moments and quantum Schatten norms.

Theorem 1.3. *Let $r \geq 2$, $\mathfrak{b}_n := \frac{nr'+d}{n+1}$ and assume K verifies*

$$\nabla K \in L^\infty + L^{\mathfrak{b},\infty} \qquad \text{for some } \mathfrak{b} \in (\mathfrak{b}_4, +\infty) \tag{1.11}$$
$$\nabla^2 K \in L^2 + L^q \qquad \text{for some } q \in (r', 2), \tag{1.12}$$

and let f be a solution of the (Vlasov) equation and $\boldsymbol{\rho}_\hbar$ be a solution of (Hartree) equation with respective initial conditions

$$f^{\text{in}} \in \mathcal{P} \cap L^\infty_{x,\xi} \text{ verifying (1.6) and (1.7)}$$
$$\boldsymbol{\rho}_\hbar^{\text{in}} \in \mathscr{P} \cap \mathcal{L}^r.$$

Assume also that the initial quantum velocity moment

$$M_{n_1}^{\text{in}} < C \text{ for a given } n_1 \geq \frac{d}{q - r'}. \tag{1.13}$$

Then there exists $T > 0$ such that for any $t \in (0, T)$,

$$W_{2,\hbar}(f(t), \boldsymbol{\rho}_\hbar(t)) \leq C_T \left(W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}_\hbar^{\text{in}}) + \sqrt{\hbar}\right).$$

Moreover, when $\mathfrak{b} \geq \frac{d+2r'}{3}$, then there exists $\Phi \in C^0(\mathbb{R}_+)$ such that for any $t > 0$

$$W_{2,\hbar}(f(t), \boldsymbol{\rho}_\hbar(t)) \leq W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}_\hbar^{\text{in}}) e^{C(t)} + C_0(t)\sqrt{\hbar}, \tag{1.14}$$

where

$$C_1(t) = \|\nabla^2 K\|^2_{L^{s,\infty}} \Phi(t)^2$$
$$C(t) = 1 + C_1(t) + \|\nabla^2 K\|_{L^q} \Phi(t)$$
$$C_0(t) = C_1(t)\, C(t)^{-1} \left(e^{2C(t)} - 1\right).$$

The next theorem proves the semi-classical convergence in a case of more singular interactions kernels such that ∇K is in the Besov space $B^1_{1,\infty}$, which includes the Coulomb potential. The definition and basic properties of Besov spaces are recalled in Appendix A.1.

Theorem 1.4. *Assume K verifies*

$$\nabla K \in L^{\mathfrak{b}} + L^\infty \qquad \text{for some } \mathfrak{b} \in \left(\tfrac{d+4}{5}, +\infty\right),$$

*and **one** of the two following conditions*

$$\nabla^2 K \in L^2 + L^q \qquad \text{for some } q \in (1,2), \tag{1.15}$$

$$\nabla K \in B^1_{1,\infty}, \tag{1.16}$$

and let f be a solution of the (Vlasov) equation and $\boldsymbol{\rho}_\hbar$ be a solution of (Hartree) equation with respective initial conditions

$$f^{\text{in}} \in \mathcal{P} \cap L^\infty_{x,\xi} \text{ verifying (1.6) and (1.7)}$$
$$\boldsymbol{\rho}_\hbar^{\text{in}} \in \mathscr{P} \cap \mathcal{L}^\infty.$$

Moreover, assume that for a given $n \in 2\mathbb{N}$ such that $n > d$

$$\forall \mathrm{i} \in [\![1,d]\!],\ \mathsf{p}_\mathrm{i}^n \boldsymbol{\rho}_\hbar^{\text{in}} \in \mathcal{L}^\infty,$$

where $\mathsf{p}_\mathrm{i} := -i\hbar\partial_\mathrm{i}$. Assume also that the initial quantum velocity moment

$$M^{\text{in}}_{n_1} < C \text{ for a given } n_1 \geq \frac{\mathfrak{b}(n-1)+d}{\mathfrak{b}-1}, \tag{1.17}$$

with $n_1 \in 2\mathbb{N}$. Then there exists $T > 0$ such that

$$M_{n_1} \in L^\infty([0,T])$$
$$\mathsf{p}_\mathrm{i}^n \boldsymbol{\rho}_\hbar \in L^\infty([0,T], \mathcal{L}^\infty) \text{ for any } \mathrm{i} \in [\![1,d]\!]$$
$$\rho_\hbar \in L^\infty([0,T], L^\infty),$$

uniformly in $\hbar$, and there exists a constant C_T depending only on the initial conditions and independent of $\hbar$ such that

$$W_{2,\hbar}(f(t), \boldsymbol{\rho}_\hbar(t)) \leq C_T \left(W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}_\hbar^{\text{in}})^{\mathrm{c}} + \sqrt{\hbar} \right),$$

with $\mathrm{c} = 1$ when (1.15) is verified and $\mathrm{c} = e^{T/\sqrt{2}}$ when (1.16) is verified. Moreover, when $\mathfrak{b} \geq \frac{d+2}{3}$ and (1.15) is verified, we can take $T = +\infty$ and the same time estimate as in Theorem 1.3 holds. If $\mathfrak{b} \geq \frac{d+2}{3}$ and (1.16) is verified (which is the case for the Coulomb potential in dimension $d = 2$), we obtain the following time dependence instead

$$W_{2,\hbar}(f(t), \boldsymbol{\rho}_\hbar(t)) \leq \max\left(\sqrt{d\hbar},\ W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}_\hbar^{\text{in}})^{e^{t/\sqrt{2}}} e^{\lambda(t)(e^{t/\sqrt{2}}-1)} \right),$$

where

$$\lambda(t) = C\left(1 + \|\nabla K\|_{B^1_{1,\infty}} \sup_{[0,t]} (\|\rho\|_{L^\infty}(t) + \|\rho_\hbar\|_{L^\infty}(t)) \right).$$

Remark 1.5. *From Theorem 1.1, we can replace the $W_{2,\hbar}$ pseudo-distance in the left of the semiclassical estimates of the two previous theorems by the classical Wasserstein distance up to adding a constant $\sqrt{2d\hbar}$. Moreover, if the initial states are superposition of coherent states, then we can also replace the $W_{2,\hbar}$ pseudo-distance in the right of the inequalities. This is detailed in Section 1.7.*

Remark 1.6. *If $K = \frac{1}{|x|^a}$ or $K = -\ln(|x|)$ if $a = 0$ (i.e. $\mathfrak{b} = \frac{d}{a+1}$) and $r = \infty$, we can summarize the results by the following table, where "global" indicates that the result is global in time and "local" that it is proved up to a fixed maximal time. We have highlighted the cases corresponding to the Coulomb interaction.*

Settings	*Moments*	*Semiclassical limit*
$d = 2$ *and* $a \in (-1, 0]$	***global***	***global***
$d = 2$ *and* $a \in \left(0, \frac{1}{2}\right]$	*global*	*?*
$d = 2$ *and* $a \in \left(\frac{1}{2}, \frac{2}{3}\right]$	*local*	*?*
$d = 3$ *and* $a \in \left(-\frac{1}{2}, \frac{4}{5}\right)$	*global*	*global*
$d = 3$ *and* $a \in \left[\frac{4}{5}, 1\right]$	***local***	***local***
$d = 3$ *and* $a \in \left(1, \frac{8}{7}\right]$	*local*	*?*
$d \geq 4$ *and* $a \in \left(\frac{d}{2} - 2, \frac{2(d-1)}{d+2}\right)$	*global*	*global*
$d \geq 4$ *and* $a \in \left[\frac{2(d-1)}{d+2}, \frac{n(d-1)}{n+d}\right]$	*local*	*local*

In particular, if $\forall \mathrm{i} \in [\![1, d]\!], \mathsf{p}_\mathrm{i}^4 \boldsymbol{\rho}_\hbar^{\mathrm{in}} \in \mathcal{L}^\infty$, it proves the convergence of the Hartree equation with Coulomb interaction potential towards the Vlasov-Poisson equation for short times in dimension $d = 3$ and all times for $d = 2$ under the assumption that M_{16}^{in} is bounded in dimension $d = 3$ and that M_8^{in} is bounded in dimension $d = 2$. As an other example, if a is close but smaller than $4/5$ in dimension $d = 3$ then (1.14) holds as soon as M_{42}^{in} is bounded.

Remark 1.7. *The hypothesis $a > \frac{d}{2} - 2$ seems harder to remove since it comes from the hypothesis $\nabla^2 K \in L^2$ which is needed for the comparison between the negative Sobolev distance and the quadratic Wasserstein distance (see Proposition 1.15).*

Remark 1.8. *As it can be seen from Proposition 1.18, the semiclassical estimate of Theorem 1.4 is actually global in time for the Coulomb potential in dimension $d = 3$ provided $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty)$. And this would follow from the propagation of order 4 velocity moments globally in time, since then Theorem 1.1 and Proposition 1.13 would imply propagation of higher order moments and weighted Lebesgue norms and the desired bound. In the classical case, global in time propagation of moments is proved in [153] through the use of a Duhamel formula in order to use the properties of dispersion of the kinetic transport semigroup. However, we did not manage to use the gain of regularity due to the dispersion. Even if it is possible to express the solution of (Hartree) through a Duhamel formula for operators, the lack of positivity of the operators involved seems to create difficulties. However, an other effect of dispersion is the decay in time of space moments which we will use in the next chapter to prove global in time estimates for small initial data.*

The rest of the chapter is organized as follows. In Section 1.2 we generalize the classical kinetic interpolation inequalities which are the key inequalities of our work. In

Section 1.3.2 we recall the conservation of energy and Schatten norms and discuss the case of interaction kernels which do not vanish at infinity.

Sections 1.4 and 1.5 prove the propagation of quantum moments (Theorem 1.1) and quantum weighted Lebesgue norms uniformly in $\hbar$ (First part of Theorem 1.4). In each case, we first write the classical version of the proof and then the quantum case which is more technical.

In Section 1.6, we prove the semiclassical limit in term of the modified Wasserstein distance using the regularity results of previous sections. It finishes the proof of Theorem 1.3 and Theorem 1.4.

Finally, Section 1.7 shows that the quantum Lebesgue norms and the quantum Wasserstein pseudo-distance are more natural when looking at superposition of coherent states. It allows us to justify more precisely the definition of the quantum Lebesgue norms and to reformulate our results in terms of the classical Wasserstein distance in this case.

1.2 Kinetic quantum interpolation inequalities

Let $n \geq 0$, $0 \leq f = f(x,\xi) \in L^r_{x,\xi} \cap L^1_{x,\xi}(|\xi|^n)$ and $\rho_f = \int_{\mathbb{R}^d} f \,\mathrm{d}\xi$. Then the classical kinetic interpolation inequality writes

$$\|\rho_f\|_{L^{p_n}} \leq C \left(\int_{\mathbb{R}^{2d}} f \,|\xi|^n \,\mathrm{d}x\,\mathrm{d}\xi\right)^{1-\theta} \|f\|^{\theta}_{L^r_{x,\xi}}, \tag{1.18}$$

where C depends only on d, n and r and $p_n' = r' + \frac{d}{n}$ and $\theta = \frac{r'}{p_n'}$ with p' denoting the Hölder conjugate of p. Even more generally, for $0 \leq k \leq n$, we have

$$\left\|\int_{\mathbb{R}^d} f \,|\xi|^k \,\mathrm{d}\xi\right\|_{L^{p_{n,k}}} \leq C \left(\int_{\mathbb{R}^{2d}} f \,|\xi|^n \,\mathrm{d}x\,\mathrm{d}\xi\right)^{1-\theta} \|f\|^{\theta}_{L^r_{x,\xi}}, \tag{1.19}$$

with $p_{n,k}' = r' + \frac{d}{n}$ and $\theta = r'/p_{n,k}'$.

The quantum version of (1.18) is known for $n = 2$ and is a variant of Lieb-Thirring inequality (see [152, (A.6)]). It reads

$$\|\rho\|_{L^p} \leq C \operatorname{Tr}(-\Delta\boldsymbol{\rho})^{1-\theta} \|\boldsymbol{\rho}\|^{\theta}_{r}, \tag{1.20}$$

with $p' = r' + \frac{d}{2}$ and $\theta = \frac{r'}{p'}$. It implies (1.18) when $n = 2$ by replacing f with $f_\hbar$, even if $f_\hbar$ is not always non-negative. Recalling the notation $\mathsf{p} = -i\hbar\nabla$ for the quantum momentum, using the $\mathcal{L}^p$ norm defined by (1.3) and remarking that

$$h^{2(1-\theta)} h^{-\theta d/r'} = h^{2-\frac{r'}{r'+d/2}\left(2+\frac{d}{r'}\right)} = 1,$$

inequality (1.20) can be written

$$\|\rho\|_{L^p} \leq C \operatorname{Tr}\!\left(|\mathsf{p}|^2\boldsymbol{\rho}\right)^{1-\theta} \|\boldsymbol{\rho}\|^{\theta}_{\mathcal{L}^r}.$$

By using the results in [87], we obtain the full generalization of (1.18).

Theorem 1.5. *Let $n \in 2\mathbb{N}$. Then there exists $C > 0$ depending only on d, r and n such that*

$$\|\rho\|_{L^p} \leq C\,\mathrm{Tr}(|\mathsf{p}|^n \boldsymbol{\rho})^{1-\theta}\,\|\boldsymbol{\rho}\|_{\mathcal{L}^r}^{\theta}, \tag{1.21}$$

with $p' = r' + \frac{d}{n}$, $\theta = \frac{r'}{p'}$. Moreover, by defining for $k \in 2\mathbb{N}$,

$$\rho_k := \sum_{j\in J} \lambda_j |\mathsf{p}^{\frac{k}{2}}\psi_j|^2 = \mathrm{diag}(\mathsf{p}^{\frac{k}{2}}\boldsymbol{\rho}\cdot\mathsf{p}^{\frac{k}{2}}),$$

for $k < n$, there exists $C > 0$ depending only on d, r, n and k such that

$$\|\rho_k\|_{L^\alpha} \leq C\,\mathrm{Tr}(|\mathsf{p}|^n \boldsymbol{\rho})^{1-\theta_k}\,\|\boldsymbol{\rho}\|_{\mathcal{L}^r}^{\theta_k}, \tag{1.22}$$

where $\alpha' = (n/k)'p'$, and $\theta_k = \frac{r'}{\alpha'}$ with $(n/k)'$ denoting the Hölder conjugate of n/k.

Remark 1.9. *Since for any $u \in \mathcal{D}'(\mathbb{R}^d,\mathbb{C})$, $\mathsf{p}u \in \mathbb{C}^d$, remark that for any $k \in \mathbb{N}$, $\mathsf{p}^k u \in \mathbb{C}^{d^k}$, which leads to $\mathsf{p}^k u = (\mathsf{p}_{\mathrm{i}_1}\ldots\mathsf{p}_{\mathrm{i}_k}u)_{(\mathrm{i}_1,\ldots,\mathrm{i}_k)\in[\![1,d]\!]^k}$ and $|\mathsf{p}^k u|$ is nothing but the natural euclidean norm on $\mathbb{C}^{d^k}$*

$$|\mathsf{p}^k u|^2 = \sum_{(\mathrm{i}_1,\ldots,\mathrm{i}_k)\in[\![1,d]\!]^k} |\mathsf{p}_{\mathrm{i}_1}\ldots\mathsf{p}_{\mathrm{i}_d}u|^2.$$

Remark 1.10. *As it can be seen in the proof, when $k = n$, we get an equality in equation (1.22)*

$$\|\rho_n\|_{L^1} = \mathrm{Tr}(|\mathsf{p}|^n\boldsymbol{\rho}) = \int_{\mathbb{R}^{2d}} f_\hbar\,|\xi|^n\,\mathrm{d}x\,\mathrm{d}\xi.$$

Remark 1.11. *Taking $\hbar = 1$, we can write (1.22) as*

$$\|\rho\|_{L^p} \leq C\,\mathrm{Tr}\Big((-\Delta)^{\frac{n}{2}}\boldsymbol{\rho}\Big)^{1-\theta}\,\|\boldsymbol{\rho}\|_r^{\theta},$$

which can be written as a Gagliardo-Nirenberg inequality for orthogonal functions under the form

$$\Bigg\|\sum_{j\in J}\lambda_j|\psi_j|^2\Bigg\|_{L^p} \leq C\left(\sum_{j\in J}\lambda_j\left\|\nabla^{\frac{n}{2}}\psi_j\right\|_{L^2}^2\right)^{1-\theta}\left(\sum_{j\in J}\lambda_j^r\|\psi_j\|_{L^2}^{2r}\right)^{\theta/r}.$$

Proof of Theorem 1.5. As proved in [87, Theorem 1], for any $s > 0$ such that $s > 1-\frac{d}{n}$, the following bound holds

$$\sum_j |\mu_j|^s \leq C_{s,n,d}\int_{\mathbb{R}^d}\mathcal{V}_-^{s+\frac{d}{n}}, \tag{1.23}$$

where the μ_j are the negative eigenvalues of $(-\Delta)^{\frac{n}{2}} + \mathcal{V}$. By taking $\mathcal{V} = -t\rho^{p-1}$ and $s = r'$, the same proof as in [152] gives inequality (1.21).

The second inequality requires some more work. We use a vector-valued version of Gagliardo-Nirenberg's inequality proved in [192] which states in particular that for a given Banach space X and any $u \in (H^n \cap L^p)(\mathbb{R}^d, X)$ we have

$$\|\nabla^{\frac{k}{2}}u\|_{L^{2\alpha}(\mathbb{R}^d,X)} \leq C_{d,k,n,p}\|u\|_{L^{2p}(\mathbb{R}^d,X)}^{1-k/n}\|\nabla^{\frac{n}{2}}u\|_{L^2(\mathbb{R}^d,X)}^{k/n}, \tag{1.24}$$

for any $(\alpha, p) \in (1,\infty]^2$, $n \in \mathbb{N}$ and $k \leq n$ such that $\frac{1}{\alpha} = \frac{1}{p}\left(1 - \frac{k}{n}\right) + \frac{k}{n}$. We will use it for the norm given for $\Psi = (\psi_j)_{j\in J}$ by

$$\|\Psi\|_X^2 := \sum_{j\in J} \lambda_j |\psi_j|^2.$$

For this norm, by integrating by parts, we remark that

$$\begin{aligned}
\|\mathsf{p}^{\frac{n}{2}}\Psi\|_{L^2(\mathbb{R}^d,X)}^2 &= \int_{\mathbb{R}^d} \sum_{j\in J} \lambda_j |\mathsf{p}^{n/2}\psi_j|^2 \\
&= \sum_{(j,\mathrm{i}_1,\dots,\mathrm{i}_{n/2})\in J\times[\![1,d]\!]^{n/2}} \lambda_j \int_{\mathbb{R}^d} \overline{\mathsf{p}_{\mathrm{i}_1}\dots\mathsf{p}_{\mathrm{i}_{n/2}}\psi_j}\,\mathsf{p}_{\mathrm{i}_1}\dots\mathsf{p}_{\mathrm{i}_{n/2}}\psi_j \\
&= \sum_{(j,\mathrm{i}_1,\dots,\mathrm{i}_{n/2})\in J\times[\![1,d]\!]^{n/2}} \lambda_j \int_{\mathbb{R}^d} \overline{\psi_j}\,\mathsf{p}_{\mathrm{i}_1}^2\dots\mathsf{p}_{\mathrm{i}_{n/2}}^2\psi_j \\
&= \sum_{j\in J} \lambda_j \int_{\mathbb{R}^d} \overline{\psi_j}|\mathsf{p}|^2\dots|\mathsf{p}|^2\psi_j \\
&= \mathrm{Tr}(|\mathsf{p}|^n\boldsymbol{\rho})\,.
\end{aligned}$$

Using inequality (1.24) for Ψ and multiplying it by $\hbar^{k/2}$, we obtain

$$\|\rho_k\|_{L^\alpha(\mathbb{R}^d,X)} \leq C_{d,k,n,p}\|\rho\|_{L^p}^{1-\frac{k}{n}}\,\mathrm{Tr}(|\mathsf{p}|^n\boldsymbol{\rho})^{k/n}\,, \tag{1.25}$$

where

$$\frac{1}{\alpha'} = \frac{1}{p'}\left(1 - \frac{k}{n}\right) = \frac{1}{p'(n/k)'}.$$

Using the first inequality (1.21) to bound $\|\rho\|_{L^p}$ in the left hand side and the fact that $\theta_k = \left(1 - \frac{k}{n}\right)\theta$, we deduce formula (1.22). □

1.3 Conservation laws

In this section, we recall the conservation laws for the (Vlasov) equation and their equivalent for (Hartree) equation.

1.3.1 Conservation of the Schatten norm

The Hamiltonian structure of the Vlasov equation implies the preservation of the Lebesgue norms

$$\|f\|_{L^r_{x,\xi}} = \|f^{\mathrm{in}}\|_{L^r_{x,\xi}}.$$

The following property is the quantum equivalent of this conservation law expressed in term of quantum Lebesgue norms.

Proposition 1.6. *Let $\boldsymbol{\rho}$ be a solution of the (Hartree) equation with initial condition $\boldsymbol{\rho}^{\mathrm{in}} \in \mathscr{P}\cap\mathcal{L}^r$. Then*

$$\|\boldsymbol{\rho}\|_{\mathcal{L}^r} = \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r}.$$

***Proof*.** Assume $r \in \mathbb{N}$. Since $\partial_t \boldsymbol{\rho} = [H_\rho, \boldsymbol{\rho}]$, we obtain

$$\begin{aligned}\partial_t \boldsymbol{\rho}^2 &= \boldsymbol{\rho}[H_\rho, \boldsymbol{\rho}] + [H_\rho, \boldsymbol{\rho}]\boldsymbol{\rho} \\ &= [H_\rho, \boldsymbol{\rho}^2],\end{aligned}$$

and by an immediate recurrence, for any $n \in \mathbb{N}$, $\partial_t \boldsymbol{\rho}^n = [H_\rho, \boldsymbol{\rho}^n]$. It implies in particular that

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Tr}(\boldsymbol{\rho}^r) = \operatorname{Tr}([H_\rho, \boldsymbol{\rho}^r]) = 0.$$

Since $\boldsymbol{\rho} \geq 0$, we can write $\boldsymbol{\rho} = |\boldsymbol{\rho}|$ and deduce that $\|\boldsymbol{\rho}\|_{\mathcal{L}^r}$ is constant in time. When r is not an integer, the result follows by complex interpolation and the case $r = +\infty$ is obtained by passing to the limit $r \to \infty$. □

1.3.2 Conservation of Energy

The conservation of energy is a well known property of both (Vlasov) and (Hartree) equations, see for example [100] and [152] for the quantum case. For the sake of completeness we write here a short proof with our notations.

Proposition 1.7. *Let $\boldsymbol{\rho} \in \mathscr{P}$ be a solution of (Hartree) equation. We define the total energy of the system by*

$$\mathcal{E}_T := M_2 + \int_{\mathbb{R}^d} \rho V,$$

where $M_2 = \operatorname{Tr}(|\mathsf{p}|^2 \boldsymbol{\rho})$ and $V = K * \rho$ for a symmetric kernel K. Then, as in the classical case, the total energy is conserved

$$\mathcal{E}_T(t) = \mathcal{E}_T(0).$$

Remark 1.12. *Notice that we can also write*

$$\mathcal{E}_T = \iint_{\mathbb{R}^{2d}} (|\xi|^2 + V(x)) f_\hbar(x,\xi) \,\mathrm{d}x \,\mathrm{d}\xi = \operatorname{Tr}\Big((|\mathsf{p}|^2 + V)\boldsymbol{\rho}\Big),$$

which shows that the energy has the same expression with the Wigner transform $f_\hbar$ as in the classical case.

Remark 1.13. *By the interpolation inequality (1.21) and assuming that M_2 is bounded and $\boldsymbol{\rho} \in \mathcal{L}^r \cap \mathcal{L}^1$, we get that $\rho \in L^p$ for $p' \in [r' + d/2, \infty]$. Thus, by the Hardy-Littlewood-Sobolev inequality, the negative part of the potential energy $(\mathcal{E}_P)_- = \int_{\mathbb{R}^d} \rho V_-$ is bounded for $K_- \in L^{\mathfrak{a},\infty} + L^\infty$ with $\mathfrak{a} = p'/2$. Therefore, if $\boldsymbol{\rho} \in \mathcal{L}^r \cap \mathcal{L}^1$ with $r' \leq 2\mathfrak{b}_0 - d/2$, $(\mathcal{E}_P)_-$ is controlled by the kinetic energy and both quantities remains finite if M_2^{in} is bounded. It includes the Coulomb interaction in dimension $d = 3$. See also [152].*

If K is not bounded for $|x| \to \infty$ but $K = K_0 + K_\infty \in L^{\mathfrak{a},\infty} + L^\infty(|x|^{-k})$, as in the case of the two-dimensional Coulomb interaction $K = -\ln(|x|)$, $\mathcal{E}_P$ can be controlled by assuming for example additional finite space moments

$$N_k = \int_{\mathbb{R}^{2d}} f_\hbar(x,\xi) |x|^k \,\mathrm{d}x \,\mathrm{d}\xi = \operatorname{Tr}\Big(|x|^k \boldsymbol{\rho}\Big) < C.$$

In this case, one can indeed write

$$\mathcal{E}_P = \int_{\mathbb{R}^d} (K_0 * \rho)\rho + \int_{\mathbb{R}^d} (K_\infty * \rho)\rho.$$

The first integral is still controlled as above by M_2 *if* $2\mathfrak{a} \in [r' + d/2, \infty]$. *to control the second, we write*

$$\begin{aligned}\left|\int_{\mathbb{R}^d} (K_\infty * \rho)\rho\right| &\leq C \int_{\mathbb{R}^d} |x-y|^k \rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\leq C \int_{\mathbb{R}^d} (|x|^k + |y|^k)\rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\leq 2CM_0N_k.\end{aligned}$$

It is easy to see that if $M_2^{\mathrm{in}} + N_2^{\mathrm{in}}$ *is bounded, then space and velocity moments up to order* 2 *remain bounded, since* $\partial_t N_2 = \mathrm{Tr}((x \cdot \mathsf{p} + \mathsf{p} \cdot x)\boldsymbol{\rho}) \leq 2N_2^{1/2}M_2^{1/2}$, *which combined with the conservation of energy leads to*

$$\begin{aligned}|\partial_t(M_2 + N_2 + \mathcal{E}_P)| &\leq M_2 + N_2 \\ &\leq M_2 + N_2 + \mathcal{E}_P + C(M_0^{\theta_1}M_2^{\theta_2} + N_0^{\theta_3}N_2^{\theta_4}). \\ &\leq (1+C)(M_2 + N_2 + \mathcal{E}_P) + CM_0.\end{aligned}$$

By Gronwall's Lemma and since $\mathcal{E}_P$ *is controlled by* $M_2 + N_2$, *we obtain that* $M_2 + N_2 \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$.

Proof of Proposition 1.7. Since $\mathrm{Tr}(H[H,\boldsymbol{\rho}]) = \mathrm{Tr}([H,H]\boldsymbol{\rho}) = 0$ and $\partial_t H = \partial_t V$, we obtain

$$\begin{aligned}2\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}(H\boldsymbol{\rho}) &= 2\,\mathrm{Tr}((\partial_t H)\boldsymbol{\rho}) + \frac{2}{i\hbar}\mathrm{Tr}(H[H,\boldsymbol{\rho}]) \\ &= 2\,\mathrm{Tr}((\partial_t V)\boldsymbol{\rho}) \\ &= 2\int_{\mathbb{R}^d} (\partial_t V)\rho,\end{aligned}$$

and since K is symmetric, we get

$$2\int_{\mathbb{R}^d} (\partial_t V)\rho = \int_{\mathbb{R}^d} (K * \partial_t \rho)\rho + (K * \rho)\partial_t\rho = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \rho V.$$

Now we remark that

$$2\,\mathrm{Tr}(H\boldsymbol{\rho}) = \mathrm{Tr}\Big(|\mathsf{p}|^2\boldsymbol{\rho}\Big) + 2\,\mathrm{Tr}(V\boldsymbol{\rho}) = M_2 + 2\int_{\mathbb{R}^d} \rho V.$$

Thus, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M_2 + 2\int_{\mathbb{R}^d} \rho V\right) = 2\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}(H\boldsymbol{\rho}) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \rho V,$$

which leads to the result. □

1.4 Propagation of moments

We study in this section the propagation independently of $\hbar$ of velocity moments for the Wigner transform of the density operator $\boldsymbol{\rho}$ solution of the (Hartree) equation, which write

$$M_n := \iint_{\mathbb{R}^{2d}} f_\hbar(x,\xi)\,|\xi|^n \,\mathrm{d}x\,\mathrm{d}\xi = \mathrm{Tr}(|\mathsf{p}|^n \boldsymbol{\rho}_\hbar)\,.$$

To clarify the presentation, we first prove the classical estimate which will be our guideline to prove the semiclassical case.

1.4.1 Classical case

In this section, we consider only the classical quantities, so that we define

$$\rho_n := \int_{\mathbb{R}^d} f(x,\xi)\,|\xi|^n \,\mathrm{d}\xi$$
$$M_n := \iint_{\mathbb{R}^{2d}} f(x,\xi)\,|\xi|^n \,\mathrm{d}x\,\mathrm{d}\xi = \int_{\mathbb{R}^d} \rho_n.$$

We can then prove the classical analogue of Theorem 1.1.

Proposition 1.8. *Let $r \geq 2$, $\mathfrak{b}_n := \frac{nr'+d}{n+1}$ and $\nabla K \in L^{\mathfrak{b},\infty}$ for a given $\mathfrak{b} \in [\mathfrak{b}_3, +\infty]$ and f verify the (Vlasov) equation for $t \in [0,T]$ with initial condition $f^{\mathrm{in}} \in \mathcal{P} \cap L^r_{x,\xi}$ such that M_0 and M_n^{in} are bounded for a given $n \geq 2$. Then there exists $T > 0$ and $\Phi \in C^0[0,T)$ such that for any $t \in [0,T)$*

$$M_n \leq \Phi(t). \tag{1.26}$$

Moreover, $T = +\infty$ when $\mathfrak{b} \geq \mathfrak{b}_2 = \frac{2r'+d}{3}$.

Proof of Proposition 1.8. Since M_0 and M_n^{in} are bounded, we deduce that M_2^{in} is bounded and by conservation of the energy (Proposition 1.7) we deduce that $M_2 \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$. To simplify we write $f = f(t,x,\xi)$. Then we have

$$\begin{aligned}\frac{\mathrm{d}M_n}{\mathrm{d}t} &= \iint_{\mathbb{R}^{2d}} (-\xi\cdot\nabla_x f - E(x)\cdot\nabla_\xi f)\,|\xi|^n \,\mathrm{d}x\,\mathrm{d}\xi \\ &= n\iint_{\mathbb{R}^{2d}} f E(x)\cdot\xi\,|\xi|^{n-2}\,\mathrm{d}x\,\mathrm{d}\xi.\end{aligned}$$

Since $E = -\nabla K * \rho$ with $\nabla K \in L^{\mathfrak{b},\infty}$, Hölder's and Hardy-Littlewood-Sobolev's inequalities give

$$\left|\frac{\mathrm{d}M_n}{\mathrm{d}t}\right| \leq n\left\|\int_{\mathbb{R}^d} f\,|\xi|^{n-1}\,\mathrm{d}\xi\right\|_{L^\alpha} \|E\|_{L^{\alpha'}} \tag{1.27}$$
$$\leq n\|\rho_{n-1}\|_{L^\alpha}\|\rho\|_{L^\beta}, \tag{1.28}$$

where $(\alpha,\beta) \in (1,\infty)^2$ are such that $1 + \frac{1}{\alpha'} = \frac{1}{\beta} + \frac{1}{\mathfrak{b}}$ or equivalently

$$\frac{1}{\alpha'} + \frac{1}{\beta'} = \frac{1}{\mathfrak{b}}.$$

By the interpolation inequality (1.19), if we can take $\alpha' = p'_{n,n-1} = np'_n = nr' + d \geq \mathfrak{b}$ and $\theta = \frac{r'}{\alpha'}$, we get

$$\|\rho_{n-1}\|_{L^\alpha} \leq C M_n^{1-\theta} \|f\|_{L^r_{x,\xi}}^\theta. \tag{1.29}$$

Moreover, since the $L^r_{x,\xi}$ is conserved, we can replace $\|f\|_{L^r_{x,\xi}}$ by $\|f^{\mathrm{in}}\|_{L^r_{x,\xi}}$. If $\beta \leq p_{n-1}$, we can bound $\|\rho\|_{L^\beta}$ using only moments of order less than $n-1$ by using the interpolation inequality (1.18)

$$\|\rho\|_{L^\beta} \leq \|\rho\|_{L^{p_{n-1}}}^{\frac{p'}{\beta'}} \|\rho\|_{L^1}^{1-\frac{p'}{\beta'}} \leq C M_0^{1-\frac{p'}{\beta'}} M_{n-1}^{\frac{p'}{\beta'}(1-\frac{r'}{p'_n})} \|f^{\mathrm{in}}\|_{L^r_{x,\xi}}^{\frac{p'r'}{\beta' p'_n}}.$$

Therefore, for $\frac{\mathrm{d}}{\mathrm{d}t} M_n$, the inequality becomes

$$\left|\frac{\mathrm{d}M_n}{\mathrm{d}t}\right| \leq C_{d,n,r} M_0^{1-\frac{p'}{\beta'}} \|f^{\mathrm{in}}\|_{L^r_{x,\xi}}^{\theta + \frac{p'r'}{\beta' p'_n}} M_{n-1}^{\frac{p'}{\beta'}(1-\frac{r'}{p'_n})} M_n^{1-\theta}.$$

Assuming that M_{n-1} is bounded on $[0,T]$, by Gronwall's Lemma, it implies a bound on $[0,T]$ for M_n.

If $\beta > p_{n-1}$, we remark that

$$\begin{aligned} \beta \leq p_n &\Leftrightarrow \frac{1}{\mathfrak{b}} - \frac{1}{\alpha'} \leq \frac{1}{p'_n} \\ &\Leftrightarrow \frac{1}{\mathfrak{b}} \leq \frac{1}{p'_n}\left(1 + \frac{1}{n}\right) \\ &\Leftrightarrow \mathfrak{b} \geq \frac{nr' + d}{n+1} =: \mathfrak{b}_n. \end{aligned}$$

In this case, by interpolation between Lebesgue spaces and by the interpolation inequality (1.18), we get

$$\begin{aligned} \|\rho\|_{L^\beta} &\leq \|\rho\|_{L^{p_n}}^\varepsilon \|\rho\|_{L^{p_{n-1}}}^{1-\varepsilon} \\ &\leq C_{d,n,r} M_{n-1}^{(1-\varepsilon)(1-\theta_{n-1})} M_n^{\varepsilon(1-\theta_n)} \|f^{\mathrm{in}}\|_{L^r_{x,\xi}}^{(1-\varepsilon)\theta_{n-1} + \varepsilon\theta_n}, \end{aligned}$$

where $\theta_n = \frac{r'}{p'_n}$ and $\varepsilon \in (0,1)$ is defined by

$$\frac{1}{\beta'} = \frac{\varepsilon}{p'_n} + \frac{1-\varepsilon}{p'_{n-1}}. \tag{1.30}$$

By (1.27) and (1.29), it implies

$$\left|\frac{\mathrm{d}M_n}{\mathrm{d}t}\right| \leq C_{d,n,r} \|f^{\mathrm{in}}\|_{L^r_{x,\xi}}^{\theta + (1-\varepsilon)\theta_{n-1} + \varepsilon\theta_n} M_{n-1}^{\Theta_0} M_n^{\Theta},$$

where

$$\begin{aligned} \Theta_0 &= (1-\varepsilon)(1-\theta_{n-1}) \\ \Theta &= 1 - \theta + \varepsilon(1-\theta_n). \end{aligned}$$

Using equation (1.30) to compute ε, we obtain

$$\begin{aligned}
\varepsilon &= \left(\frac{1}{\beta'} - \frac{1}{p'_{n-1}}\right)\left(\frac{1}{p'_n} - \frac{1}{p'_{n-1}}\right)^{-1} \\
&= \left(\frac{1}{\mathfrak{b}} - \frac{1}{np'_n} - \frac{1}{p'_{n-1}}\right)\left(\frac{1}{p'_n} - \frac{1}{p'_{n-1}}\right)^{-1} \\
&= \frac{n(n-1)}{d}\left(\frac{p'_n p'_{n-1}}{\mathfrak{b}} - \frac{p'_{n-1}}{n} - p'_n\right) \\
&= \frac{(nr'+d)((n-1)r'+d)}{d\mathfrak{b}} - \frac{(n-1)r'}{d}(1+n) - n.
\end{aligned}$$

Since $1 - \frac{r'}{p'_n} = \frac{d}{d+nr'}$ we deduce that

$$\begin{aligned}
\Theta &= 1 - \frac{r'}{np'_n} + \varepsilon\frac{d}{d+nr'} \\
&= \frac{d+(n-1)r'}{d+nr'} + \frac{((n-1)r'+d)}{\mathfrak{b}} - \frac{(n-1)r'}{d+nr'}(1+n) - \frac{nd}{d+nr'} \\
&= \frac{(n-1)r'+d}{\mathfrak{b}} - n + 1 = 1 + n\left(\frac{\mathfrak{b}_{n-1}}{\mathfrak{b}} - 1\right).
\end{aligned}$$

In particular,

$$\Theta \leq 1 \Leftrightarrow \mathfrak{b} \geq \frac{(n-1)r'+d}{n} = \mathfrak{b}_{n-1}, \tag{1.31}$$

which, by Gronwall's Lemma, allows to prove that M_n is bounded on $[0,T]$ when M_{n-1} is bounded $[0,T]$ and M_n^{in} is bounded. In particular, since M_2 is bounded by the energy conservation and $\mathfrak{b} = 1 + \frac{d-r'}{n}$ is decreasing with n, all the moments will be propagated if the moment of order 3 is bounded, which is the case when $\mathfrak{b} \geq \frac{d+2r'}{3}$. □

1.4.2 Quantum case

As we will not consider the (Vlasov) equation in this section, we will omit to write the $\hbar$ for $\boldsymbol{\rho}_\hbar$ solution of (Hartree) equation and $\rho = \operatorname{diag}(\boldsymbol{\rho}_\hbar)$ to simplify the notations.

Proof of Theorem 1.1. To simplify the computations, we define for any $k \in \mathbb{N}$,

$$\begin{aligned}
[\mathsf{p}]^{2k} &:= |\mathsf{p}|^{2k} \\
[\mathsf{p}]^{2k+1} &:= |\mathsf{p}|^{2k}\mathsf{p}.
\end{aligned}$$

Step 1. An inequality for the time derivative of moments. We remark that

$$\begin{aligned}
[\mathsf{p}, H] &= [\mathsf{p}, V] = -i\hbar\nabla(V\cdot) + i\hbar V\nabla = i\hbar E \\
[|\mathsf{p}|^2, H] &= \mathsf{p}\cdot\mathsf{p}H - H\mathsf{p}\cdot\mathsf{p} = \mathsf{p}\cdot[\mathsf{p}, H] + [\mathsf{p}, H]\cdot\mathsf{p} = i\hbar(\mathsf{p}\cdot E + E\cdot\mathsf{p}) \\
[|\mathsf{p}|^{2n+2}, H] &= |\mathsf{p}|^2\left[|\mathsf{p}|^{2n}, H\right] + \left[|\mathsf{p}|^{2n}, H\right]|\mathsf{p}|^2.
\end{aligned}$$

By an immediate recurrence, we deduce that for any $n \in \mathbb{N}$,

$$\frac{1}{i\hbar}[|\mathsf{p}|^{2n+2}, H] = \sum_{k=0}^{n} \binom{n}{k} |\mathsf{p}|^{2k}(\mathsf{p} \cdot E + E \cdot \mathsf{p})|\mathsf{p}|^{2(n-k)}. \tag{1.32}$$

With this formula, we can compute the time derivative of moments as follows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Tr}\Big(|\mathsf{p}|^{2n+2}\boldsymbol{\rho}\Big) &= \sum_{k=0}^{n} \binom{n}{k} \operatorname{Tr}\Big(|\mathsf{p}|^{2k}(\mathsf{p} \cdot E + E \cdot \mathsf{p})|\mathsf{p}|^{2(n-k)}\boldsymbol{\rho}\Big) \\
&= \sum_{k=0}^{n} \binom{n}{k} \operatorname{Tr}\Big(([\mathsf{p}]^{2k+1} \cdot E[\mathsf{p}]^{2(n-k)} + [\mathsf{p}]^{2k}E \cdot [\mathsf{p}]^{2(n-k)+1})\boldsymbol{\rho}\Big) \\
&= \sum_{k=0}^{2n+1} \binom{n}{\lfloor k/2 \rfloor} \operatorname{Tr}\Big([\mathsf{p}]^{k} \cdot E \cdot [\mathsf{p}]^{2n+1-k}\boldsymbol{\rho}\Big). \qquad (1.33)
\end{aligned}$$

Recalling that $\boldsymbol{\rho} = \sum_{j\in J} \lambda_j |\psi_j\rangle\langle\psi_j|$, for the term with $k = n$, we have

$$\begin{aligned}
\operatorname{Tr}\Big([\mathsf{p}]^{n} \cdot E \cdot [\mathsf{p}]^{n+1}\boldsymbol{\rho}\Big) &= \sum_{j\in J} \lambda_j \int_{\mathbb{R}^d} ([\mathsf{p}]^n\overline{\psi_j}) \cdot E \cdot ([\mathsf{p}]^{n+1}\psi_j) \\
&\leq \int_{\mathbb{R}^d} |E| \rho_{2n}^{\frac{1}{2}} \rho_{2n+2}^{\frac{1}{2}}.
\end{aligned}$$

For the other terms, for $k < n$, we integrate by parts and use the Cauchy-Schwarz inequality to find

$$\begin{aligned}
\operatorname{Tr}\Big([\mathsf{p}]^{k} \cdot E \cdot [\mathsf{p}]^{2n+1-k}\boldsymbol{\rho}\Big) &= \sum_{j\in J} \lambda_j \int_{\mathbb{R}^d} [\mathsf{p}]^{n-k}(E[\mathsf{p}]^k\overline{\psi_j})([\mathsf{p}]^{n+1}\psi_j) \qquad (1.34) \\
&\leq \int_{\mathbb{R}^d} \left(\sum_{j\in J} \lambda_j \left|[\mathsf{p}]^{n-k}(E[\mathsf{p}]^k\psi_j)\right|^2\right)^{\frac{1}{2}} \rho_{2n+2}^{\frac{1}{2}}.
\end{aligned}$$

Next we use the definition of E to write

$$\sum_{j\in J} \lambda_j \left|[\mathsf{p}]^{n-k}(E[\mathsf{p}]^k\psi_j)\right|^2 = \sum_{j\in J} \lambda_j \left|\sum_{j_2\in J} \lambda_{j_2}[\mathsf{p}]^{n-k}\left(\left(\nabla K * |\psi_{j_2}|^2\right)[\mathsf{p}]^k\psi_j\right)\right|^2. \tag{1.35}$$

To continue, we introduce the multi-index notation

$$\begin{aligned}
a &= (a_{\mathrm{i}})_{\mathrm{i}\in[\![1,d]\!]} \in \mathbb{N}^d \text{ is a finite sequence of integers} \\
|a| &= \sum_{\mathrm{i}=1}^{d} a_{\mathrm{i}} \\
\mathsf{p}^a &= (-i\hbar)^{|a|}\partial_{x_1}^{a_1}\partial_{x_2}^{a_2}\dots\partial_{x_d}^{a_d}.
\end{aligned}$$

With these notations, we can write

$$|\mathsf{p}|^{2n}(uv) = \sum_{|a+b|=2n} C_{a,b}^{2n}\, \mathsf{p}^a(u)\mathsf{p}^b(v),$$

where the constants $C^{2n}_{a,b}$ are non-negative integers depending on the multi-indices a and b and such that

$$\sum_{|a+b|=2n} C^{2n}_{a,b} \leq (4d)^n. \tag{1.36}$$

More generally, we will write

$$[\mathsf{p}]^n(uv) = \widetilde{\sum_{|a+b|=n}} C^n_{a,b}\, \mathsf{p}^a(u)\mathsf{p}^b(v),$$

where the sum is taken only over the (a,b) such that $|a+b| = n-1$ if n is odd, since then $[\mathsf{p}]^n(uv)$ is a vector with one free index. Hence, we get

$$[\mathsf{p}]^{n-k}\left(\left(\nabla K * |\psi_{j_2}|^2\right)[\mathsf{p}]^k\psi_j\right) = \widetilde{\sum_{|a+b+c|=n-k}} C^{n-k}_{a,b,c} \nabla K * \left(\mathsf{p}^a(\overline{\psi_{j_2}})\mathsf{p}^b(\psi_{j_2})\right)\mathsf{p}^c[\mathsf{p}]^k\psi_j.$$

Moreover, by the Cauchy-Schwarz inequality

$$\begin{aligned}\left|\sum_{j_2\in J}\lambda_{j_2}\mathsf{p}^a(\overline{\psi_{j_2}})\mathsf{p}^b(\psi_{j_2})\right| &\leq \left(\sum_{j\in J}\lambda_j|\mathsf{p}^a(\psi_j)|^2\right)^{\frac12}\left(\sum_{j_2\in J}\lambda_{j_2}|\mathsf{p}^b(\psi_{j_2})|^2\right)^{\frac12}\\ &\leq \rho^{1/2}_{2|a|}\rho^{1/2}_{2|b|}.\end{aligned}$$

Thus, (1.35) leads to the following inequality

$$\sum_{j\in J}\lambda_j\left|[\mathsf{p}]^{n-k}(E[\mathsf{p}]^k\psi_j)\right|^2 \leq \sum_{j\in J}\lambda_j\left|\widetilde{\sum_{|a+b+c|=n-k}} C^{n-k}_{a,b,c}\left(|\nabla K| * \left(\rho^{1/2}_{2|a|}\rho^{1/2}_{2|b|}\right)\right)|\mathsf{p}^c[\mathsf{p}]^k\psi_j|\right|^2.$$

The left hand side can be written under the form

$$\left\|\widetilde{\sum_{|a+b+c|=n-k}} A_{a,b,c}\Psi_c\right\|^2_X,$$

with $\Psi_c = (|\mathsf{p}^c[\mathsf{p}]^k\psi_j|)_{j\in J}$ and $\|(u_j)_{j\in J}\|^2_X = \sum_{j\in J}\lambda_j|u_j|^2$. Then, Minkowski's inequality reads

$$\left\|\widetilde{\sum_{|a+b+c|=n-k}} A_{a,b,c}\Psi_c\right\|_X \leq \widetilde{\sum_{|a+b+c|=n-k}} A_{a,b,c}\left\|\Psi_c\right\|_X.$$

Remarking that $\|\Psi_c\|^2_X \leq \rho_{2|c|+2k}$, we obtain

$$\sum_{j\in J}\lambda_j\left|[\mathsf{p}]^{n-k}(E[\mathsf{p}]^k\psi_j)\right|^2 \leq \left(\widetilde{\sum_{|a+b+c|=n-k}} C^{n-k}_{a,b,c}\left(|\nabla K| * \left(\rho^{1/2}_{2|a|}\rho^{1/2}_{2|b|}\right)\right)\rho^{1/2}_{2|c|+2k}\right)^2.$$

Combining this inequality with (1.34) and (1.36), we obtain

$$\begin{aligned}\mathrm{Tr}\Big([\mathsf{p}]^k\cdot E\cdot[\mathsf{p}]^{2n+1-k}\boldsymbol{\rho}\Big) &\leq \int_{\mathbb{R}^d}\widetilde{\sum_{|a+b+c|=n-k}} C^{n-k}_{a,b,c}\left(\nabla K * \left(\rho^{1/2}_{2|a|}\rho^{1/2}_{2|b|}\right)\right)\rho^{\frac12}_{2|c|+2k}\rho^{\frac12}_{2n+2}\\ &\leq (4d)^{\frac{n-k}{2}}\sup_{|a+b+c|=n}\left\|\nabla K * \left(\rho^{1/2}_{2|a|}\rho^{1/2}_{2|b|}\right)\rho^{\frac12}_{2|c|}\right\|_{L^2}\|\rho_{2n+2}\|^{\frac12}_{L^1}\\ &\leq (4d)^{\frac{n-k}{2}}C_K\sup_{|a+b+c|=n}\left\|\rho_{2|a|}\right\|^{\frac12}_{L^\alpha}\left\|\rho_{2|b|}\right\|^{\frac12}_{L^\beta}\left\|\rho_{2|c|}\right\|^{\frac12}_{L^\gamma}M^{\frac12}_{2n+2},\end{aligned}$$

where $C_K = \|\nabla K\|_{L^{\mathfrak{b},\infty}}$,

$$\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} = \frac{2}{\mathfrak{b}}, \tag{1.37}$$

and we used Hölder's inequality and the weak Young's inequality. The case $k > n$ is treated in the same way. Thus, from (1.33) and the identity

$$\begin{aligned}\sum_{k=0}^{2n+1} \binom{n}{\lfloor k/2 \rfloor} (4d)^{\frac{n-k}{2}} &= \left(2\sqrt{d}\right)^n \left(\sum_{k=0}^{n} \binom{n}{k} \left(2\sqrt{d}\right)^{-(2k+1)} + \sum_{k=0}^{n} \binom{n}{k} \left(2\sqrt{d}\right)^{-2k}\right) \\ &= \frac{\left(1+2\sqrt{d}\right)(1+4d)^n}{(4d)^{\frac{n+1}{2}}} =: c_{d,2n+2},\end{aligned}$$

we deduce that for any $n \in \mathbb{N}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Tr}\!\left(|\mathsf{p}|^{2n+2}\boldsymbol{\rho}\right) \leq c_{d,2n+2} C_K \sup_{|a+b+c|=n} \left\|\rho_{2|a|}\right\|_{L^\alpha}^{\frac12} \left\|\rho_{2|b|}\right\|_{L^\beta}^{\frac12} \left\|\rho_{2|c|}\right\|_{L^\gamma}^{\frac12} M_{2n+2}^{\frac12}.$$

Step 2. Using the kinetic interpolation. To simplify the notations, we will fix $n \in 2\mathbb{N}$ and write previous formula as

$$\frac{\mathrm{d}M_n}{\mathrm{d}t} \leq c_{d,n} C_K M_n^{\frac12} \sup_{|a+b+c|=n/2-1} \left\|\rho_{2|a|}\right\|_{L^\alpha}^{\frac12} \left\|\rho_{2|b|}\right\|_{L^\beta}^{\frac12} \left\|\rho_{2|c|}\right\|_{L^\gamma}^{\frac12}. \tag{1.38}$$

To bound the right term by powers of M_n, we use the kinetic quantum interpolation inequalities (1.22), which gives for any $k \in \{2|a|, 2|b|, 2|c|\} \subset [\![0, n-2]\!]$

$$\|\rho_k\|_{L^{p_n(k)}} \leq C_{d,r,n,k} M_n^{1-\theta_n(k)} \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^{\theta_n(k)}, \tag{1.39}$$

where $p_n'(k) = (n/k)' p_n'$ with $p_n' = r' + \frac{d}{n}$ and $\theta_n(k) = \frac{r'}{p_n'(k)}$. Since $k \leq n-2$ the same inequality holds by replacing n by $n-2$. If we can choose $\alpha, \beta, \gamma > 1$ and $\varepsilon \in (0,1)$ such that

$$\frac{1}{\alpha'} = \frac{\varepsilon}{p_n'(2|a|)} + \frac{1-\varepsilon}{p_{n-2}'(2|a|)} \tag{1.40}$$

$$\frac{1}{\beta'} = \frac{\varepsilon}{p_n'(2|b|)} + \frac{1-\varepsilon}{p_{n-2}'(2|b|)} \tag{1.41}$$

$$\frac{1}{\gamma'} = \frac{\varepsilon}{p_n'(2|c|)} + \frac{1-\varepsilon}{p_{n-2}'(2|c|)}. \tag{1.42}$$

By interpolation and since by Proposition 1.6, $\|\boldsymbol{\rho}\|_{\mathcal{L}^r} = \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r}$, we get

$$\begin{aligned}\|\rho_{2|a|}\|_{L^\alpha} &\leq \|\rho_{2|a|}\|_{L^{p_n(2|a|)}}^{\varepsilon} \|\rho_{2|a|}\|_{L^{p_{n-2}(2|a|)}}^{1-\varepsilon} \\ &\leq C_{d,r,n,|a|} \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r}^{C_{a,n,\varepsilon}} M_{n-2}^{(1-\varepsilon)(1-\theta_{n-2}(2|a|))} M_n^{\varepsilon(1-\theta_n(2|a|))}.\end{aligned}$$

Since $|a+b+c| = n/2-1$, we remark that

$$\begin{aligned}
\frac{1}{2}\left(\frac{1}{p'_{n-2}(2|a|)}+\frac{1}{p'_{n-2}(2|b|)}+\frac{1}{p'_{n-2}(2|c|)}\right) &= \frac{1}{2p'_{n-2}}\left(3-2\frac{|a|+|b|+|c|}{n-2}\right)\\
&= \frac{1}{p'_{n-2}} = \frac{n-2}{(n-2)r'+d}\\
\frac{1}{\mathfrak{b}_n} := \frac{1}{2}\left(\frac{1}{p'_n(2|a|)}+\frac{1}{p'_n(2|b|)}+\frac{1}{p'_n(2|c|)}\right) &= \frac{1}{2p'_n}\left(3-2\frac{|a|+|b|+|c|}{n}\right)\\
&= \frac{1}{p'_n}\left(1+\frac{1}{n}\right) = \frac{n+1}{nr'+d}.
\end{aligned}$$

Therefore, by (1.37), we get

$$\frac{1}{\mathfrak{b}} = \frac{\varepsilon}{\mathfrak{b}_n} + \frac{1-\varepsilon}{p'_{n-2}}. \tag{1.43}$$

Let first assume that $\mathfrak{b} \leq p'_{n-2}$. Then, since by assumption $\mathfrak{b} \geq \mathfrak{b}_n$, we can find $(\alpha,\beta,\gamma,\varepsilon)$ verifying (1.40), (1.41) and (1.42). Hence, (1.38) becomes

$$\frac{\mathrm{d}M_n}{\mathrm{d}t} \leq C_{d,r,n} C_K \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r}^{\Theta_2} M_{n-2}^{\Theta_0} M_n^{\frac{1}{2}+\Theta_1}, \tag{1.44}$$

with

$$\begin{aligned}
\Theta_1 &= \frac{\varepsilon}{2}\left(3-\theta_n(a)-\theta_n(b)-\theta_n(c)\right) = \varepsilon\left(\frac{3}{2}-\frac{r'}{\mathfrak{b}_n}\right)\\
\Theta_0 &= (1-\varepsilon)\left(\frac{3}{2}-\frac{r'}{p'_{n-2}}\right)\\
\Theta_2 &= \frac{3}{2}-\Theta_1-\Theta_0.
\end{aligned}$$

From (1.43), we can compute ε and we get

$$\varepsilon = \frac{nr'+d}{(n-2)r'+3d}\left(\frac{(n-2)r'+d}{\mathfrak{b}}-(n-2)\right).$$

It leads to the following formula for $\Theta = 1/2+\Theta_1$

$$\Theta = 1+\frac{d+(n-2)r'}{2}\left(\frac{1}{\mathfrak{b}}-\frac{n-1}{d+(n-2)r'}\right) = 1+\frac{n-1}{2}\left(\frac{\mathfrak{b}_{n-2}}{\mathfrak{b}}-1\right).$$

In particular,

$$\begin{aligned}
&\Theta \leq 1 \;\Leftrightarrow \mathfrak{b}_{n-2} \leq \mathfrak{b}\\
&\Theta \underset{n\to\infty}{\to} 1.
\end{aligned}$$

The result then follows by Grönwall's Lemma. If $\mathfrak{b} > p'_{n-2}$, it is no more possible to write (1.43), but we can still find $\tilde{\varepsilon} \in (0,1)$ such that

$$\frac{1}{\mathfrak{b}} = \frac{1-\tilde{\varepsilon}}{\tilde{\mathfrak{b}}} + \tilde{\varepsilon}\left(\frac{\varepsilon}{\mathfrak{b}_n}+\frac{1-\varepsilon}{p'_{n-2}}\right),$$

where

$$\frac{1}{\tilde{\mathfrak{b}}} = \frac{1}{p'_{2|a|}(2|a|)} + \frac{1}{p'_{2|b|}(2|b|)} + \frac{1}{p'_{2|c|}(2|c|)} = 0,$$

and we obtain

$$\frac{\mathrm{d}M_n}{\mathrm{d}t} \leq C_n(M_{2|a|}, M_{2|b|}, M_{2|c|}) M_{n-2}^{\tilde{\varepsilon}\Theta_0} M_n^{\frac{1}{2}+\tilde{\varepsilon}\Theta_1}, \tag{1.45}$$

with $\tilde{\Theta} = 1/2 + \tilde{\varepsilon}\Theta_1 \leq \Theta$ and we can again conclude by Gronwall's Lemma. □

1.5 Propagation of higher Lebesgue weighted norms

1.5.1 Classical case

As previously, we first do the proof in the classical case as a guideline for the proof of the quantum case. The goal here is to propagate $\|f\|_{L^p_{x,\xi}(|\xi|^n)}$ norms uniformly in p. Together with the uniform bound on $\|f\|_{L^p_{x,\xi}}$, it leads to the following bound for some $C, T > 0$ and any $t \in [0, T]$

$$0 \leq f(t, x, \xi) \leq \frac{C}{1 + |\xi|^n}.$$

For $n > d$, this bound implies that $\rho := \int_{\mathbb{R}^d} f \,\mathrm{d}\xi \in L^\infty([0, T] \times \mathbb{R}^d)$.

Proposition 1.9. *Assume $E \in L^\infty([0, T], L^\infty)$ and let f be a solution of the (Vlasov) equation such that $f^{\mathrm{in}} \in L^p$ and $f^{\mathrm{in}}|\xi|^n \in L^p$ for a given $p \in [1, \infty]$. Then*

$$\|f|\xi|^n\|_{L^p_{x,\xi}} \leq \left(\left\| f^{\mathrm{in}}|\xi|^n \right\|_{L^p_{x,\xi}}^{\frac{1}{n}} + \|E\|_{L^\infty} \left\| f^{\mathrm{in}} \right\|_{L^p_{x,\xi}}^{\frac{1}{n}} t \right)^n.$$

Corollary 1.10. *Assume f verifies the hypothesis of Proposition 1.9 for $n > d$ and $p = \infty$. Then $\rho \in L^\infty((0, T), L^\infty)$ and*

$$\begin{aligned} \|\rho\|_{L^\infty} &\leq C \|f\|_{L^\infty_{x,\xi}(1+|\xi|^n)} \\ &\leq \left(\left\| f^{\mathrm{in}}|\xi|^n \right\|_{L^p_{x,\xi}}^{\frac{1}{n}} + \|E\|_{L^\infty} \left\| f^{\mathrm{in}} \right\|_{L^\infty_{x,\xi}}^{\frac{1}{n}} t \right)^n + \|f^{\mathrm{in}}\|_{L^\infty}. \end{aligned}$$

Proof. Since $f = f(t, x, v)$ is solution of the (Vlasov) equation, differentiating with respect to time and integrating by parts, we get

$$\begin{aligned} \frac{1}{p} \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}^{2d}} |f|\xi|^n|^p \,\mathrm{d}x \,\mathrm{d}\xi &= \iint_{\mathbb{R}^{2d}} |f|^{p-2} f \left(-\xi \cdot \nabla_x f - E(x) \cdot \nabla_\xi f \right) |\xi|^{np} \,\mathrm{d}x \,\mathrm{d}\xi \\ &= \frac{1}{p} \iint_{\mathbb{R}^{2d}} \left(-\xi \cdot \nabla_x \left(|f|^p \right) - E(x) \cdot \nabla_\xi \left(|f|^p \right) \right) |\xi|^{np} \,\mathrm{d}x \,\mathrm{d}\xi \\ &= n \iint_{\mathbb{R}^{2d}} |f|^p E(x) \cdot \xi \, |\xi|^{np-2} \,\mathrm{d}x \,\mathrm{d}\xi. \end{aligned}$$

Using the fact that $E \in L^\infty$ and Hölder's inequality, we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\left\|f|\xi|^n\right\|_{L^p_{x,\xi}}^p \leq n\|E\|_{L^\infty}\iint_{\mathbb{R}^{2d}}|f|^p\,|\xi|^{np-1}\,\mathrm{d}x\,\mathrm{d}\xi$$
$$\leq n\|E\|_{L^\infty}\left\|f|\xi|^n\right\|_{L^p_{x,\xi}}^{p-\frac{1}{n}}\|f\|_{L^p_{x,\xi}}^{\frac{1}{n}}.$$

This inequality can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|f|\xi|^n\right\|_{L^p_{x,\xi}} \leq n\|E\|_{L^\infty}\left\|f|\xi|^n\right\|_{L^p_{x,\xi}}^{1-\frac{1}{n}}\|f\|_{L^p_{x,\xi}}^{\frac{1}{n}}.$$

Then by conservation of the $L^p_{x,\xi}$ norm and Gronwall's Lemma, we deduce that

$$\left\|f|\xi|^n\right\|_{L^p_{x,\xi}} \leq \left(\left\|f^{\mathrm{in}}|\xi|^n\right\|_{L^p_{x,\xi}}^{\frac{1}{n}} + \|E\|_{L^\infty}\left\|f^{\mathrm{in}}\right\|_{L^p_{x,\xi}}^{\frac{1}{n}}t\right)^n,$$

and if $\|f\|_{L^\infty_{x,\xi}} < \infty$ and $\|f^{\mathrm{in}}|\xi|^n\|_{L^\infty_{x,\xi}} < \infty$, we can pass to the limit $p \to \infty$. □

1.5.2 Quantum case

In this section, we again only focus on the quantum objects, so that we will write $\boldsymbol{\rho} := \boldsymbol{\rho}_\hbar$ and $\rho := \operatorname{diag}(\boldsymbol{\rho})$ to simplify the notations. For $k \in \mathbb{R}_+$, we define the $\mathcal{L}(|\mathsf{p}|^k)$ space as the space of compact operators $\boldsymbol{\rho}$ such that

$$\|\boldsymbol{\rho}\|_{\mathcal{L}^p(|\mathsf{p}|^k)} := \left\||\mathsf{p}|^k\boldsymbol{\rho}\right\|_{\mathcal{L}^p} < C,$$

where $\mathcal{L}^p$ is defined by (1.3). Remark that if $\boldsymbol{\rho}$ is self-adjoint, then $|\boldsymbol{\rho}|\mathsf{p}|^k|^2 = |\mathsf{p}|^k|\boldsymbol{\rho}|^2|\mathsf{p}|^k$ and by cyclicity of the trace, for any $p \in 2\mathbb{N}$,

$$\|\boldsymbol{\rho}\|_{\mathcal{L}^p(|\mathsf{p}|^k)} = \left\||\boldsymbol{\rho}|\mathsf{p}|^k\right\|_{\mathcal{L}^p}.$$

Actually, as proved in [89], this is true also for $p = 1$ and can be easily generalized to any $p \in \mathbb{R}_+$, since for any self-adjoint compact operators A and B, as pointed out in [199, Formula (1.3)], the singular values are the same for AB and $(AB)^* = BA$, which leads to

$$\|AB\|_p = \|BA\|_p. \tag{1.46}$$

We recall Hölder's inequality (see e.g. [199, Theorem 2.8]) which reads for any compact operators A and B

$$\|AB\|_r \leq \|A\|_p\|B\|_q \text{ when } \frac{1}{p}+\frac{1}{q}=\frac{1}{r}, \tag{1.47}$$

and the Araki-Lieb-Thirring inequality [8, Theorem 1] which reads

$$\operatorname{Tr}((BAB)^{qr}) \leq \operatorname{Tr}((B^qA^qB^q)^r), \tag{1.48}$$

for any operators $A, B \geq 0$ and $(q, r) \in [1, \infty) \times \mathbb{R}_+$. Remark that for $A, B \geq 0$, since $|AB| = (BA^2B)^{\frac{1}{2}}$, we can rewrite (1.48) as

$$\|AB\|_{qr}^q \leq \|A^qB^q\|_r \text{ for any } q \geq 1. \tag{1.49}$$

From these inequalities we deduce the following interpolation inequality

Proposition 1.11. *Let $A \geq 0$ be a compact operator, then for any $\theta \in [0,1]$*

$$\|AB^\theta\|_p \leq \|AB\|_p^\theta \|A\|_p^{1-\theta}.$$

Proof. Since $A \geq 0$, we can write $A = A^\theta A^{1-\theta}$ and by Hölder's inequality (1.47), we obtain

$$\begin{aligned}\left\|AB^\theta\right\|_p &\leq \left\|A^\theta B^\theta\right\|_{p/\theta} \left\|A^{1-\theta}\right\|_{p/(1-\theta)} \\ &\leq \left\|A^\theta B^\theta\right\|_{p/\theta} \|A\|_p^{1-\theta}.\end{aligned}$$

Then, we use (1.49) with $q = 1/\theta \geq 1$ to get

$$\left\|B^\theta A^\theta\right\|_{p/\theta} \leq \|AB\|_p^\theta,$$

which proves the result. □

As a corollary of the previous proposition, taking $B = |\mathsf{p}|^n$ and $A = \boldsymbol{\rho}$, we obtain results for the $\mathcal{L}(|\mathsf{p}|^k)$ norm.

Corollary 1.12. *Let $\boldsymbol{\rho}$ be a non-negative hermitian operator, then for any $0 \leq k \leq n < \infty$*

$$\|\boldsymbol{\rho}\|_{\mathcal{L}^p(|\mathsf{p}|^k)} \leq \|\boldsymbol{\rho}\|_{\mathcal{L}^p(|\mathsf{p}|^n)}^{k/n} \|\boldsymbol{\rho}\|_{\mathcal{L}^p}^{1-k/n}. \tag{1.50}$$

We are now ready to prove the propagation of weighted quantum Schatten norms.

Proposition 1.13. *Let $\mathsf{p}_\mathrm{i} := -i\hbar\partial_\mathrm{i}$ for a given $\mathrm{i} \in [\![1,d]\!]$,*

$$\nabla K \in L^{\mathfrak{b}} + L^\infty \qquad\qquad \textit{for some } \mathfrak{b} \in (1,+\infty),$$

$r \in (\mathfrak{b}', \infty]$ and $\boldsymbol{\rho} \in \mathscr{P} \cap \mathcal{L}^r$ verify (Hartree) equation. Assume moreover that M_{n_1} is bounded on $[0,T]$ for a given $T > 0$ and a given $n_1 \in \mathbb{N}$ and that $\boldsymbol{\rho}^{\mathrm{in}} \in \mathcal{L}^{2p}(\mathsf{p}_\mathrm{i}^n)$ for a given $p \in \mathbb{N} \cup \{\infty\}$ such that $2p \leq r$ and a given $n \in \mathbb{N}$ such that

$$n \leq \theta\, n_1 + 1 - \frac{d}{\mathfrak{b}}, \tag{1.51}$$

with $\theta = 1 - \frac{r'}{\mathfrak{b}}$. Then for any $t \in [0,T]$,

$$\|\boldsymbol{\rho}\|_{\mathcal{L}^{2p}(\mathsf{p}_\mathrm{i}^n)} \leq 2^n \left(\|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^{2p}(\mathsf{p}_\mathrm{i}^n)} + \tilde{C}_{\boldsymbol{\rho}^{\mathrm{in}}} \left(t + \int_0^t M_{n_1}^\theta\right)^n\right), \tag{1.52}$$

where $\tilde{C}_{\boldsymbol{\rho}^{\mathrm{in}}} = (4^n C_{d,r,n_1} \|\nabla K\|_{L^{\mathfrak{b}}}(1+M_0))^n \, \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^{2p}}^{1+\frac{nr'}{\mathfrak{b}}}$. In particular, for $r = p = \infty$, we obtain

$$\|\boldsymbol{\rho}\|_{\mathcal{L}^\infty(\mathsf{p}_\mathrm{i}^n)} \leq 2^n \left(\|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^\infty(\mathsf{p}_\mathrm{i}^n)} + \tilde{C}_{\boldsymbol{\rho}^{\mathrm{in}}} \left(t + \int_0^t M_{n_1}^\theta\right)^n\right), \tag{1.53}$$

with $\tilde{C}_{\boldsymbol{\rho}^{\mathrm{in}}} = (4^n C_{d,n_1} \|\nabla K\|_{L^{\mathfrak{b}}}(1+M_0))^n \, \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^\infty}^{1+\frac{n}{\mathfrak{b}}}$.

Corollary 1.14. *With the hypotheses of Proposition 1.13, assume that for a given $n > d/2$, $\boldsymbol{\rho}^{\rm in} \in \mathcal{L}^\infty(\mathsf{p}_{\rm i}^{2n})$ for all $\rm i \in [\![1,d]\!]$. Then*

$$\|\rho\|_{L^\infty} \leq c_{d,n} \|\boldsymbol{\rho}\|_{L^\infty((0,T),\mathcal{L}^\infty(1+\mathsf{p}^{2n}))}, \tag{1.54}$$

which is bounded independently from $\hbar$.

Proof of Corollary 1.14. To prove (1.54), we remark that from Proposition 1.13,

$$P_n\boldsymbol{\rho} := \left(1 + \sum_{\rm i=1}^d \mathsf{p}_{\rm i}^{2n}\right)\boldsymbol{\rho} \in L^\infty([0,T],\mathcal{L}^\infty).$$

Since P_n and ρ are non-negative self-adjoint operators, by using (1.49) for $r = \infty$ and $q = 2$, we obtain

$$\|P_n^{\frac{1}{2}}\boldsymbol{\rho}P_n^{\frac{1}{2}}\|_\infty = \|\boldsymbol{\rho}^{\frac{1}{2}}P_n^{\frac{1}{2}}\|_\infty^2 \leq \|P_n\boldsymbol{\rho}\|_\infty \leq C_\rho h^d,$$

where $C_\rho = \|P_n\boldsymbol{\rho}\|_{L^\infty((0,T),\mathcal{L}^\infty)}$. From this, we get that for any $\varphi \in L^2$,

$$\langle\varphi|P_n^{\frac{1}{2}}\boldsymbol{\rho}P_n^{\frac{1}{2}}\varphi\rangle \leq C_\rho h^d\|\varphi\|^2 = \langle\varphi|C_\rho h^d\varphi\rangle,$$

or equivalently $P_n^{\frac{1}{2}}\boldsymbol{\rho}P_n^{\frac{1}{2}} \leq C_\rho h^d$. It implies that $A := C_\rho h^d - P_n^{\frac{1}{2}}\boldsymbol{\rho}P_n^{\frac{1}{2}}$ is a non-negative self-adjoint operator. Using the Fourier transform, we remark that P_n is invertible and that for any $\varphi \in L^2$ we have

$$\begin{aligned} P_n^{-1}\varphi(x) &= \mathcal{F}_y\left(\frac{\hat{\varphi}}{1+\sum_{\rm i=1}^d |hy_{\rm i}|^{2n}}\right)(x) \\ &= \int_{\mathbb{R}^d} \mathcal{F}_y\left(\frac{1}{1+\sum_{\rm i=1}^d |hy_{\rm i}|^{2n}}\right)(x-z)\varphi(z)\mathrm{d}z. \end{aligned}$$

Since $P_n^{-\frac{1}{2}}$ is a positive operator, we deduce that $C_\rho h^d P_n^{-1} - \boldsymbol{\rho} = P_n^{-\frac{1}{2}}AP_n^{-\frac{1}{2}}$ is a non-negative operator of diagonal

$$\begin{aligned} 0 \leq k(x,x) &= C_\rho h^d\,\mathcal{F}_y\left(\frac{1}{1+\sum_{\rm i=1}^d |hy_{\rm i}|^{2n}}\right)(0) - \rho(x) \\ &= C_\rho h^d \int_{\mathbb{R}^d} \frac{1}{1+\sum_{\rm i=1}^d |hy_{\rm i}|^{2n}}\mathrm{d}y - \rho(x) \\ &= c_{d,n}C_\rho - \rho(x), \end{aligned}$$

where, since $2n > d$,

$$c_{d,n} := \int_{\mathbb{R}^d} \frac{\mathrm{d}x}{1+\sum_{\rm i=1}^d |x_{\rm i}|^{2n}} < \infty.$$

Since $\rho \geq 0$, we deduce that

$$0 \leq \rho(x) \leq c_{d,n}C_\rho,$$

which proves the result. ☐

Proof of Proposition 1.13. By cyclicity of the trace, for $p \in \mathbb{N}$, $\mathrm{i} \in [\![1, d]\!]$ and $n \in \mathbb{N}$ we have

$$\mathrm{Tr}\Big(|\mathsf{p}_\mathrm{i}^n \boldsymbol{\rho}|^{2p}\Big) = \mathrm{Tr}\Big(\big(\mathsf{p}_\mathrm{i}^{2n} \boldsymbol{\rho}^2\big)^p\Big)\,.$$

Therefore, using again the cyclicity of the trace

$$\begin{aligned}\frac{i\hbar}{p}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}\Big(|\mathsf{p}_\mathrm{i}^n \boldsymbol{\rho}|^{2p}\Big) &= \mathrm{Tr}\Big(\big(\mathsf{p}_\mathrm{i}^{2n} \boldsymbol{\rho}^2\big)^{p-1}\, \mathsf{p}_\mathrm{i}^{2n}[H, \boldsymbol{\rho}^2]\Big)\\ &= \mathrm{Tr}\Big(\big(\mathsf{p}_\mathrm{i}^{2n} \boldsymbol{\rho}^2\big)^{p-1}\, \mathsf{p}_\mathrm{i}^{2n} H \boldsymbol{\rho}^2\Big) - \mathrm{Tr}\Big(\big(\mathsf{p}_\mathrm{i}^{2n} \boldsymbol{\rho}^2\big)^{p-1}\, \mathsf{p}_\mathrm{i}^{2n} \boldsymbol{\rho}^2 H\Big)\\ &= \mathrm{Tr}\Big(\mathsf{p}_\mathrm{i}^{2n} H \boldsymbol{\rho}^2 \,\big(\mathsf{p}_\mathrm{i}^{2n} \boldsymbol{\rho}^2\big)^{p-1}\Big) - \mathrm{Tr}\Big(H \mathsf{p}_\mathrm{i}^{2n} \boldsymbol{\rho}^2 \,\big(\mathsf{p}_\mathrm{i}^{2n} \boldsymbol{\rho}^2\big)^{p-1}\Big)\\ &= \mathrm{Tr}\Big([\mathsf{p}_\mathrm{i}^{2n}, H] \boldsymbol{\rho}^2 \,\big(\mathsf{p}_\mathrm{i}^{2n} \boldsymbol{\rho}^2\big)^{p-1}\Big)\\ &= \mathrm{Tr}\Big(\boldsymbol{\rho}[\mathsf{p}_\mathrm{i}^{2n}, H]\boldsymbol{\rho}\, |\mathsf{p}_\mathrm{i}^n \boldsymbol{\rho}|^{2p-2}\Big)\,.\end{aligned}$$

Now we write $[\mathsf{p}_\mathrm{i}^{2n}, H]$ in terms of E thanks to formula (1.32) to obtain in the same way

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}\Big(|\mathsf{p}_\mathrm{i}^n \boldsymbol{\rho}|^{2p}\Big) &= \sum_{k=0}^{n-1} \binom{n-1}{k} \mathrm{Tr}\Big(\boldsymbol{\rho}\mathsf{p}_\mathrm{i}^{2(n-1-k)}(\mathsf{p}_\mathrm{i} E_\mathrm{i} + E_\mathrm{i}\mathsf{p}_\mathrm{i})\mathsf{p}_\mathrm{i}^{2k}\boldsymbol{\rho} P\Big)\\ &= \sum_{k=1}^{2n} \binom{n-1}{\lfloor (k-1)/2 \rfloor} \mathrm{Tr}\Big(\boldsymbol{\rho}\mathsf{p}_\mathrm{i}^{2n-k} E_\mathrm{i} \mathsf{p}_\mathrm{i}^{k-1}\boldsymbol{\rho} P\Big)\,, \end{aligned} \tag{1.55}$$

where $P = |\mathsf{p}_\mathrm{i}^n \boldsymbol{\rho}|^{2p-2}$. Since $\mathrm{Tr}(A^*) = \overline{\mathrm{Tr}(A)}$, the following holds

$$\mathrm{Tr}\Big(\boldsymbol{\rho}\mathsf{p}_\mathrm{i}^{2n-k} E_\mathrm{i} \mathsf{p}_\mathrm{i}^{k-1}\boldsymbol{\rho} P\Big) = \overline{\mathrm{Tr}\Big(\boldsymbol{\rho}\mathsf{p}_\mathrm{i}^{k-1} E_\mathrm{i} \mathsf{p}_\mathrm{i}^{2n-k}\boldsymbol{\rho} P\Big)},$$

so that (1.55) becomes

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}\Big(|\mathsf{p}_\mathrm{i}^n \boldsymbol{\rho}|^{2p}\Big) = 2\Re \sum_{k=1}^{n} \binom{n-1}{\lfloor (k-1)/2 \rfloor} \mathrm{Tr}\Big(\boldsymbol{\rho}\mathsf{p}_\mathrm{i}^{2n-k} E_\mathrm{i} \mathsf{p}_\mathrm{i}^{k-1}\boldsymbol{\rho} P\Big)\,. \tag{1.56}$$

To treat the right term, we remark that Leibniz rule for differentiation leads to

$$\mathsf{p}_\mathrm{i}^{n-k} E_\mathrm{i} = \sum_{m=0}^{n-k} \binom{n-k}{m} (\mathsf{p}_\mathrm{i}^m(E_\mathrm{i}))\mathsf{p}_\mathrm{i}^{n-k-m}.$$

Therefore we obtain

$$\mathrm{Tr}\Big(\boldsymbol{\rho}\mathsf{p}_\mathrm{i}^{2n-k} E_\mathrm{i} \mathsf{p}_\mathrm{i}^{k-1}\boldsymbol{\rho} P\Big) = \sum_{m=0}^{n-k} \binom{n-k}{m} \mathrm{Tr}\Big(\boldsymbol{\rho}\mathsf{p}_\mathrm{i}^{n}(\mathsf{p}_\mathrm{i}^m(E_\mathrm{i}))\mathsf{p}_\mathrm{i}^{n-m-1}\boldsymbol{\rho} P\Big)\,. \tag{1.57}$$

Thus we can use Hölder's inequality (1.47) and the interpolation inequality (1.50) to get

$$\begin{aligned}&|\,\mathrm{Tr}(\boldsymbol{\rho}\mathsf{p}_\mathrm{i}^n\,(\mathsf{p}_\mathrm{i}^m(E_\mathrm{i}))\mathsf{p}_\mathrm{i}^{n-m-1}\boldsymbol{\rho} P)|\\ &\qquad= \Big|\mathrm{Tr}\Big((\mathsf{p}_\mathrm{i}^m(E_\mathrm{i}))\mathsf{p}_\mathrm{i}^{n-m-1}\boldsymbol{\rho} P \boldsymbol{\rho}\mathsf{p}_\mathrm{i}^n\Big)\Big|\\ &\qquad\leq \|\mathsf{p}_\mathrm{i}^m(E_\mathrm{i})\|_\infty \left\|\mathsf{p}_\mathrm{i}^{n-m-1}\boldsymbol{\rho}\right\|_{2p} \left\||\mathsf{p}_\mathrm{i}^n \boldsymbol{\rho}|^{2p-2}\right\|_{\frac{2p}{(2p-2)}} \|\mathsf{p}_\mathrm{i}^n \boldsymbol{\rho}\|_{2p}\\ &\qquad\leq \|\mathsf{p}_\mathrm{i}^m(E_\mathrm{i})\|_{L^\infty} \|\boldsymbol{\rho}\|_{2p}^{\frac{m+1}{n}} \|\mathsf{p}_\mathrm{i}^n \boldsymbol{\rho}\|_{2p}^{2p-\frac{m+1}{n}}\,.\end{aligned} \tag{1.58}$$

The term $\|\boldsymbol{\rho}\|_{2p}$ will be controlled by propagation of the $\mathcal{L}^p$ norm (see Proposition 1.6). To control $\|\mathsf{p}_{\mathrm{i}}^m(E_{\mathrm{i}})\|_{L^\infty}$ for any $m \in [\![0, n-1]\!]$, by interpolation, it is sufficient to prove that it is bounded for $m = 0$ and $m = n-1$. We use again the Leibniz rule to get

$$\begin{aligned} -\mathsf{p}_{\mathrm{i}}^m(E_{\mathrm{i}}) &= \mathsf{p}_{\mathrm{i}}^m\left(\nabla K * \rho\right) \\ &= \nabla K * \sum_j \lambda_j \mathsf{p}_{\mathrm{i}}^m(|\psi_j|^2) \\ &= \sum_{l=0}^m \binom{m}{l} \nabla K * \sum_j \lambda_j \mathsf{p}_{\mathrm{i}}^l(\overline{\psi_j})\mathsf{p}_{\mathrm{i}}^{m-l}(\psi_j). \end{aligned}$$

Therefore, by Hölder's inequality, recalling the notation

$$\rho_{2k} := \sum \lambda_j \left|\mathsf{p}^k\psi_j\right|^2,$$

we get the following bound

$$\begin{aligned} \|\mathsf{p}_{\mathrm{i}}^m(E_{\mathrm{i}})\|_{L^\infty} &\leq \sum_{l=0}^m \binom{m}{l} \left\| |\nabla K| * \left(\rho_{2l}^{1/2}\rho_{2(m-l)}^{1/2}\right)\right\|_{L^\infty} \\ &\leq C_K \sum_{l=0}^m \binom{m}{l} \|\rho_{2l}\|_{L^{q_1}}^{1/2} \|\rho_{2(m-l)}\|_{L^{q_2}}^{1/2}, \end{aligned} \tag{1.59}$$

where $C_K = \|\nabla K\|_{L^{\mathfrak{b}}}$ and

$$\frac{1}{q_1'} + \frac{1}{q_2'} = \frac{2}{\mathfrak{b}}. \tag{1.60}$$

From the hypothesis (1.51) for n, we get $n < n_1 + 1$ and

$$(n_1 + 1 - n)\mathfrak{b} \geq (n_1 r' + d)\,.$$

By defining $p_{n_1,k}' := (n_1/k)' p_{n_1}' = (n_1/k)'\,(r' + d/n_1)$, it implies that

$$\mathfrak{b} \geq p_{n_1,n-1}'.$$

Moreover, by the interpolation inequalities (1.22) and the fact that M_{n_1} and M_0 are bounded on $[0,T]$, we deduce that $\|\rho_k\|_{L^p}$ is bounded uniformly with respect to $\hbar$ for any $t \in [0,T]$ and any $p \in [1, p_{n_1,k}]$. In particular, since

$$\frac{2}{\mathfrak{b}} \leq \frac{2}{p_{n_1,n-1}'} \leq \frac{2}{p_{n_1,m}'} = \frac{1}{p_{n_1,2l}'} + \frac{1}{p_{n_1,2(m-l)}'},$$

we can find $q_1, q_2 \geq 1$ such that the left hand side of (1.59) is bounded on $[0,T]$ and (1.60) is verified, and there exists $(\varepsilon_1, \varepsilon_2) \in (0,1)^2$ such that

$$\begin{aligned} \frac{1}{q_1'} &= \frac{\varepsilon_1}{p_{n_1,2l}'} \\ \frac{1}{q_2'} &= \frac{\varepsilon_2}{p_{n_1,2(m-l)}'} \\ \|\rho_{2l}\|_{L^{q_1}} \|\rho_{2(m-l)}\|_{L^{q_2}} &\leq \|\rho_{2l}\|_{L^{p_{n_1,2l}}}^{\varepsilon_1} \|\rho_{2(m-l)}\|_{L^{p_{n_1,2(m-l)}}}^{\varepsilon_2} \|\rho_{2l}\|_{L^1}^{1-\varepsilon_2} \|\rho_{2(m-l)}\|_{L^1}^{1-\varepsilon_2} \\ &\leq C_{d,r,n_1}^2 \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^{2\Theta_0} M_{n_1}^{2\Theta_1} M_{2l}^{1-\varepsilon_1} M_{2(m-l)}^{1-\varepsilon_2}, \end{aligned} \tag{1.61}$$

where

$$\Theta_0 = \frac{1}{2}\left(\varepsilon_1\left(\frac{r'}{p'_{n_1,2l}}\right) + \varepsilon_2\left(\frac{r'}{p'_{n_1,2(m-l)}}\right)\right)$$
$$\Theta_1 = \frac{1}{2}\left(\varepsilon_1\left(1-\frac{r'}{p'_{n_1,2l}}\right) + \varepsilon_2\left(1-\frac{r'}{p'_{n_1,2(m-l)}}\right)\right).$$

Since by (1.60), $\frac{2}{\mathfrak{b}} = \frac{\varepsilon_1}{p'_{n_1,2l}} + \frac{\varepsilon_2}{p'_{n_1,2(m-l)}}$, we deduce that

$$\Theta_0 = \frac{r'}{\mathfrak{b}}$$
$$\Theta_1 = \frac{1}{2}(\varepsilon_1+\varepsilon_2) - \frac{r'}{\mathfrak{b}}.$$

Moreover, by interpolation, for any $k \in [0, n_1]$,

$$M_k \leq M_{n_1}^{k/n_1} M_0^{1-k/n_1} \leq M_0 + M_{n_1}.$$

Using this inequality for $k = 2l$ and $k = 2(m-l)$ in (1.61), inequality (1.59) becomes

$$\|\mathsf{p}_{\mathrm{i}}^m(E_{\mathrm{i}})\|_{L^\infty} \leq 2^m C_{\boldsymbol{\rho}^{\mathrm{in}}}\left(1 + M_{n_1}^\theta\right), \tag{1.62}$$

where $\theta = 1 - \frac{r'}{\mathfrak{b}}$, $C_{\boldsymbol{\rho}^{\mathrm{in}}} = C_{d,r,n_1} C_K \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r}^{\Theta_0}(1+M_0)$ and we used the propagation of the $\mathcal{L}^r$ and $\mathcal{L}^1$ norm (Proposition 1.6). We can now come back to (1.56). By combining it with (1.57), (1.58) and (1.62), we arrive at

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathsf{p}_{\mathrm{i}}^n\boldsymbol{\rho}\|_{2p}\right) &= \frac{1}{2p\,\|\mathsf{p}_{\mathrm{i}}^n\boldsymbol{\rho}\|_{2p}^{2p-1}}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}\left(|\mathsf{p}_{\mathrm{i}}^n\boldsymbol{\rho}|^{2p}\right) \\
&\leq C_{\boldsymbol{\rho}^{\mathrm{in}}}\left(1+M_{n_1}^\theta\right)\sum_{k=1}^{n}\sum_{m=0}^{n-k}\binom{n-1}{\lfloor (k-1)/2\rfloor}\binom{n-k}{m}2^m\,\|\boldsymbol{\rho}\|_{2p}^{\frac{m+1}{n}}\,\|\mathsf{p}_{\mathrm{i}}^n\boldsymbol{\rho}\|_{2p}^{1-\frac{m+1}{n}} \\
&\leq 4^n C_{\boldsymbol{\rho}^{\mathrm{in}}}\left(1+M_{n_1}^\theta\right)\left(\|\boldsymbol{\rho}\|_{2p}^{\frac{1}{n}}\,\|\mathsf{p}_{\mathrm{i}}^n\boldsymbol{\rho}\|_{2p}^{1-\frac{1}{n}} + \|\boldsymbol{\rho}\|_{2p}\right).
\end{aligned}$$

By Multiplying the inequality by $h^{-d/(2p)'}$ and by conservation of the $\mathcal{L}^{2p}$ norm, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathsf{p}_{\mathrm{i}}^n\boldsymbol{\rho}\|_{\mathcal{L}^{2p}} \leq 4^n C_{\boldsymbol{\rho}^{\mathrm{in}}}\left(1+M_{n_1}^\theta\right)\left(\left\|\boldsymbol{\rho}^{\mathrm{in}}\right\|_{\mathcal{L}^{2p}}^{\frac{1}{n}}\|\mathsf{p}_{\mathrm{i}}^n\boldsymbol{\rho}\|_{\mathcal{L}^{2p}}^{1-\frac{1}{n}} + \left\|\boldsymbol{\rho}^{\mathrm{in}}\right\|_{\mathcal{L}^{2p}}\right).$$

Defining $u := \frac{\|\mathsf{p}_{\mathrm{i}}^n\boldsymbol{\rho}\|_{\mathcal{L}^{2p}}}{\|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^{2p}}}$ and $c(t) := 4^n C_{\boldsymbol{\rho}^{\mathrm{in}}}\int_0^t\left(1+M_{n_1}^\theta\right)$, it can be written

$$\frac{\mathrm{d}u}{\mathrm{d}t} \leq \left(1+u^{1-\frac{1}{n}}\right)\frac{\mathrm{d}c}{\mathrm{d}t}.$$

By Grönwall's Lemma, we obtain

$$\begin{aligned}
u(t) &\leq (2c(t)+u(0)) + (u(0)^{\frac{1}{n}} + 2c(t)/n)^n \\
&\leq 2^n(u(0)+c(t)^n),
\end{aligned}$$

or equivalently

$$\|\mathsf{p}_{\mathrm{i}}^n \boldsymbol{\rho}\|_{\mathcal{L}^{2p}} \leq 2^n \left(\left\|\mathsf{p}_{\mathrm{i}}^n \boldsymbol{\rho}^{\mathrm{in}}\right\|_{\mathcal{L}^{2p}} + \tilde{C}_{\boldsymbol{\rho}^{\mathrm{in}}} \left(t + \int_0^t M_{n_1}^\theta \right)^n \right), \tag{1.63}$$

where $\tilde{C}_{\boldsymbol{\rho}^{\mathrm{in}}} = \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^{2p}} (4^n C_{\boldsymbol{\rho}^{\mathrm{in}}})^n$. It proves inequality (1.52). Remark that if $r = \infty$, then we can take $C_{\boldsymbol{\rho}^{\mathrm{in}}}$ depending only on $\boldsymbol{\rho}^{\mathrm{in}}$ and not on p since by interpolation between $\mathcal{L}^p$ spaces (Proposition 1.47), we have

$$\left\|\boldsymbol{\rho}^{\mathrm{in}}\right\|_{\mathcal{L}^{2p}} \leq \left\|\boldsymbol{\rho}^{\mathrm{in}}\right\|_{\mathcal{L}^\infty}^{1/(2p)'} \left\|\boldsymbol{\rho}^{\mathrm{in}}\right\|_{\mathcal{L}^1}^{1/(2p)} \leq \left\|\boldsymbol{\rho}^{\mathrm{in}}\right\|_{\mathcal{L}^\infty} + \left\|\boldsymbol{\rho}^{\mathrm{in}}\right\|_{\mathcal{L}^1}.$$

Therefore we can pass to the limit $p \to \infty$ in (1.63) to get

$$\|\mathsf{p}_{\mathrm{i}}^n \boldsymbol{\rho}\|_{\mathcal{L}^\infty} \leq 2^n \left(\left\|\mathsf{p}_{\mathrm{i}}^n \boldsymbol{\rho}^{\mathrm{in}}\right\|_{\mathcal{L}^\infty} + \tilde{C}_{\boldsymbol{\rho}^{\mathrm{in}}} \left(t + \int_0^t M_{n_1}^\theta \right)^n \right),$$

with $\tilde{C}_{\boldsymbol{\rho}^{\mathrm{in}}} = 4^{n^2} C_{d,n_1}^n \|\nabla K\|_{L^{\mathfrak{b}}}^n \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^\infty}^{1+n/\mathfrak{b}} (1 + M_0)^n$. □

1.6 The quantum coupling estimate

Following the ideas of Loeper in [157], we use the property of displacement convexity of the interpolation between probability measures induced by the optimal transport to deduce the following bound in Wasserstein distance.

Proposition 1.15. *Let $p \in [1, +\infty]$ and $(\rho_0, \rho_1) \in (L^p \cap \mathcal{P}(\mathbb{R}^d))^2$. Then*

$$\|\rho_0 - \rho_1\|_{\dot{W}^{-1,\frac{2p}{p+1}}} \leq \max(\|\rho_0\|_{L^p}, \|\rho_1\|_{L^p})^{\frac{1}{2}} W_2(\rho_0, \rho_1), \tag{1.64}$$

where $\dot{W}^{-1,r}$ denotes the dual space of the space

$$\dot{W}^{1,r'} := \left\{ \varphi, \nabla\varphi \in L^{r'}, \varphi \underset{|x|\to\infty}{\longrightarrow} 0 \right\}.$$

Proof. Let $q = p'$ be the Hölder conjugate of p, T be the optimal transport map for the W_2 distance and $\varphi \in \dot{W}^{1,2q}$. Then the interpolant $\rho_\theta = ((1-\theta)x + \theta T(x))_\# \rho_0$ verifies

$$\int_{\mathbb{R}^d} \varphi \rho_\theta = \int_{\mathbb{R}^d} \varphi(x_\theta) \rho_0(\mathrm{d}x),$$

where we denote by $x_\theta := (1-\theta)x + \theta T(x)$. By differentiating with respect to θ and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta} \int_{\mathbb{R}^d} \varphi \rho_\theta &= \int_{\mathbb{R}^d} (T(x) - x) \cdot \nabla\varphi(x_\theta) \rho_0(\mathrm{d}x) \\ &\leq \left(\int_{\mathbb{R}^d} |T(x) - x|^2 \rho_0(\mathrm{d}x) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |\nabla\varphi|^2 \rho_\theta \right)^{\frac{1}{2}}. \end{aligned}$$

The first integral is nothing but the W_2 distance between ρ_0 and ρ_1. Thus, using Hölder's inequality to bound the second integral, we get

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\mathbb{R}^d}\varphi\rho_\theta \leq W_2(\rho_0,\rho_1)\|\varphi\|_{\dot{W}^{1,2q}}\|\rho_\theta\|_{L^p}^{1/2}.$$

By displacement convexity (see for example [191, Proposition 7.29]), the following inequality holds

$$\|\rho_\theta\|_{L^p} \leq \max(\|\rho_0\|_{L^p}, \|\rho_1\|_{L^p}).$$

Noticing that $(2q)' = \frac{2p}{p+1}$, an integration with respect to θ on $[0,1]$ gives the expected result. □

As a consequence of Proposition 1.15 and the weak Young inequality, we get the following inequality

Corollary 1.16. *Let $p \in (1,+\infty]$, $s = (2p)'$ and K be such that $\nabla^2 K \in L^{s,\infty}$. Then, we have*

$$\|\nabla K * (\rho_0 - \rho_1)\|_{L^2} \leq \|\nabla^2 K\|_{L^{s,\infty}} \max(\|\rho_0\|_{L^p}, \|\rho_1\|_{L^p})^{\frac{1}{2}} W_2(\rho_0,\rho_1). \tag{1.65}$$

If $p = 1$, the same formula holds by replacing $L^{2,\infty}$ by L^2 and if $p = \infty$ by replacing $L^{1,\infty}$ by L^1.
Moreover, if $p = \infty$, $\|\nabla^2 K\|_{L^1}$ can be replaced by $\|\nabla K\|_{B^1_{1,\infty}}$.

Proof. Let $r = \frac{2p}{p+1}$. We first write that for $\rho := \rho_0 - \rho_1$ and $\varphi \in L^2$,

$$\left|\int_{\mathbb{R}^d}(\nabla K * \rho)\varphi\right| \leq \|\rho\|_{\dot{W}^{-1,r}}\|\nabla K * \varphi\|_{\dot{W}^{1,r'}}. \tag{1.66}$$

Then, as a consequence of the weak Young inequality (see e.g. [148, Chapter 4, (7)]), we have

$$\|\nabla K * \varphi\|_{\dot{W}^{1,r'}} = \|\nabla^2 K * \varphi\|_{L^{r'}} \leq \|\varphi\|_{L^2}\|\nabla^2 K\|_{L^{s,\infty}}, \tag{1.67}$$

with $\frac{1}{s} = 1 + \frac{1}{r'} - \frac{1}{2} = 1 - \frac{1}{2p}$. Combining (1.66) and (1.67), by duality, we deduce

$$\|\nabla K * \rho\|_{L^2} \leq \|\rho\|_{W^{-1,r}}\|\nabla^2 K\|_{L^{s,\infty}}.$$

We then use Proposition 1.15 to conclude. When $p = \infty$ and $r = 2$, we use the fact that

$$\|\nabla K * \varphi\|_{\dot{H}^1} \leq \|\varphi\|_{L^2}\|\nabla K\|_{B^1_{1,\infty}}, \tag{1.68}$$

which is proved in Appendix (see (A.6) in Proposition A.16). □

We can now prove the following key estimate in the modified Wasserstein distance as defined by (1.4).

Proposition 1.17. *Let* $(s,q) \in (1,2) \times [1,\infty]$ *and assume*

$$\nabla^2 K \in L^{s,\infty} \cap L^q,$$

with $L^{s,\infty}$ *replaced by* L^2 *if* $s = 2$. *Let* $\boldsymbol{\rho}_\hbar \in \mathscr{P}$ *be a solution of* (Hartree) *equation and* f *be a solution of the* (Vlasov) *equation such that the spatial densities verify*

$$\rho_\hbar := \int_{\mathbb{R}^d} \tilde{f}_\hbar \,\mathrm{d}\xi \in L^\infty([0,T], L^{q'} \cap L^{s'/2})$$
$$\rho := \int_{\mathbb{R}^d} f \,\mathrm{d}\xi \in L^\infty([0,T], L^\infty),$$

uniformly with respect to $\hbar$. *Then, for all* $t \in [0,T]$, *we have*

$$W_{2,\hbar}(f(t), \boldsymbol{\rho}_\hbar(t)) \leq W_{2,\hbar}(f^{\mathrm{in}}, \boldsymbol{\rho}_\hbar^{\mathrm{in}}) e^{Ct} + C_0(t)\sqrt{\hbar},$$

where

$$C_1 = \|\nabla^2 K\|_{L^{s,\infty}+L^2} \sup_{[0,T]} \Big(\max(\|\rho\|_{L^{s'/2}}, \|\rho_\hbar\|_{L^{s'/2}})^{1/2} \|\rho\|_{L^\infty}^{1/2} \Big)$$
$$C = 1 + C_1 + \sup_{[0,T]} \|\rho_\hbar\|_{L^{q'}} \|\nabla^2 K\|_{L^q}$$
$$C_0(t) = C_1 \sqrt{d} (C)^{-1} (e^{Ct} - 1).$$

Proof. Let $p = q'$ and $\tilde{p} = s'/2$. As in [105, Section 4], we define the time dependent coupling $\boldsymbol{\gamma}(z) = \boldsymbol{\gamma}_\hbar(t,z)$ with $z = (x,\xi)$ as the solution to the Cauchy problem

$$\partial_t \boldsymbol{\gamma} = \{H, \boldsymbol{\gamma}\} + \frac{1}{i\hbar}[H_\hbar, \boldsymbol{\gamma}],$$

with initial condition $\boldsymbol{\gamma}^{\mathrm{in}} \in \mathcal{C}(f^{\mathrm{in}}, \boldsymbol{\rho}_\hbar^{\mathrm{in}})$. As proved in [105, Lemma 4.2], $\boldsymbol{\gamma} \in \mathcal{C}(f(t), \boldsymbol{\rho}_\hbar(t))$. We also define

$$\mathcal{E}_\hbar = \mathcal{E}_\hbar(t) := \int_{\mathbb{R}^{2d}} \mathrm{Tr}(\mathbf{c}_\hbar(z)\boldsymbol{\gamma}(z)) \,\mathrm{d}z.$$

By differentiating in time, we get

$$\frac{\mathrm{d}\mathcal{E}_\hbar}{\mathrm{d}t} = \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(\Big(\{H, \mathbf{c}_\hbar(z)\} + \frac{1}{i\hbar}[H_\hbar, \mathbf{c}_\hbar] \Big) \boldsymbol{\gamma}(z) \Big) \,\mathrm{d}z,$$

which, by a direct computation, as detailed in [105, Section 4.3], leads to

$$\frac{\mathrm{d}\mathcal{E}_\hbar}{\mathrm{d}t} \leq \mathcal{E}_\hbar + \int_{\mathbb{R}^{2d}} \mathrm{Tr}_y((\xi - \mathsf{p}) \cdot (E_\hbar(y) - E(x))\boldsymbol{\gamma}(z)) \,\mathrm{d}z + \int_{\mathbb{R}^{2d}} \mathrm{Tr}_y((E_\hbar(y) - E(x)) \cdot (\xi - \mathsf{p})\boldsymbol{\gamma}(z)) \,\mathrm{d}z. \tag{1.69}$$

Since $\boldsymbol{\gamma} \geq 0$, we use the fact that by Hölder's inequality for Schatten spaces (see e.g. [199]) and cyclicity of the trace, we have for any operators $(A,B) \in \mathcal{L}(L^2, L^2(\mathbb{R}^d, \mathbb{R}^d))^2$

$$\begin{aligned} \mathrm{Tr}(A^* B \boldsymbol{\gamma})^2 &= \mathrm{Tr}\Big(\boldsymbol{\gamma}^{1/2} A^* B \boldsymbol{\gamma}^{1/2}\Big)^2 \\ &\leq \mathrm{Tr}\Big(|\boldsymbol{\gamma}^{1/2} A^*|^2\Big) \mathrm{Tr}\Big(|B\boldsymbol{\gamma}^{1/2}|^2\Big) \\ &\leq \mathrm{Tr}(A\boldsymbol{\gamma}A^*) \,\mathrm{Tr}\Big(\boldsymbol{\gamma}^{1/2} B^* B \boldsymbol{\gamma}^{1/2}\Big) \\ &\leq \mathrm{Tr}\Big(|A|^2\boldsymbol{\gamma}\Big) \,\mathrm{Tr}\Big(|B|^2\boldsymbol{\gamma}\Big). \end{aligned}$$

Thus, using this inequality for $A = (\xi - \mathsf{p})$ and $B = E_\hbar(y) - E(x)$ for the first integral in (1.69) and $A = E_\hbar(y) - E(x)$ and $B = (\xi - \mathsf{p})$ for the second integral, we get by the Cauchy-Schwarz inequality

$$\frac{\mathrm{d}\mathcal{E}_\hbar}{\mathrm{d}t} \leq \mathcal{E}_\hbar + 2\left(\int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(|\xi - \mathsf{p}|^2 \boldsymbol{\gamma}(z)\Big)\,\mathrm{d}z\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(|E_\hbar(y) - E(x)|^2 \boldsymbol{\gamma}(z)\Big)\,\mathrm{d}z\right)^{\frac{1}{2}}. \quad (1.70)$$

The first integral is bounded by $\mathcal{E}_\hbar$ and second integral by $2(I_1 + I_2)$ where

$$I_1 = \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(|E_\hbar(x) - E(x)|^2 \boldsymbol{\gamma}(z)\Big)\,\mathrm{d}z$$
$$I_2 = \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(|E_\hbar(y) - E_\hbar(x)|^2 \boldsymbol{\gamma}(z)\Big)\,\mathrm{d}z.$$

Then, since $\boldsymbol{\gamma} \in \mathcal{C}(f, \boldsymbol{\rho}_\hbar)$, by corollary 1.16, we can control I_1 in the following way

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^d} |\nabla K * (\rho_\hbar - \rho)(x)|^2 \rho(x)\,\mathrm{d}x \\ &\leq \|\nabla^2 K\|_{L^{s,\infty}}^2 \max(\|\rho\|_{L^{\tilde{p}}}, \|\rho_\hbar\|_{L^{\tilde{p}}}) W_2(\rho, \rho_\hbar)^2 \ \|\rho\|_{L^\infty}. \end{aligned}$$

Moreover, since $\rho_\hbar = \int_{\mathbb{R}^d} \tilde{f}_\hbar(t,x,\xi)\,\mathrm{d}\xi$ is nothing but the projection of $f_\hbar$ on the space of positions, we have $W_2(\rho, \rho_\hbar) \leq W_2(f, \tilde{f}_\hbar)$ (see Proposition A.17 for a more detailed proof). Using Theorem 1.1 and the definition of $W_{2,\hbar}$, we get

$$\begin{aligned} I_1 &\leq C_1^2\, W_2(f, \tilde{f}_\hbar)^2 \\ &\leq C_1^2 \left(W_{2,\hbar}(f, \boldsymbol{\rho}_\hbar)^2 + d\hbar\right) \\ &\leq C_1^2 \left(\mathcal{E}_\hbar + d\hbar\right). \end{aligned} \quad (1.71)$$

In order to control I_2, we remark that, from Young's inequality, we get

$$\|\nabla E_\hbar\|_{L^\infty} = \|\nabla^2 K * \rho_\hbar\|_{L^\infty} \leq \|\rho_\hbar\|_{L^p} \|\nabla^2 K\|_{L^q},$$

which implies that $E_\hbar \in C^{0,1}$ uniformly with respect to $\hbar$, and

$$I_2 \leq C_2^2 \int_{\mathbb{R}^{2d}} \mathrm{Tr}\Big(|y - x|^2 \boldsymbol{\gamma}_\hbar(t,z)\Big)\,\mathrm{d}z \leq C_2^2 \mathcal{E}_\hbar,$$

where $C_2 = \|\rho_\hbar\|_{L^p} \|\nabla^2 K\|_{L^q}$. By combining this estimate with (1.71), equation (1.70) becomes

$$\begin{aligned} \frac{\mathrm{d}\mathcal{E}_\hbar}{\mathrm{d}t} &\leq \mathcal{E}_\hbar + \sqrt{\mathcal{E}_\hbar}\left(2(C_1^2 + C_2^2)\mathcal{E}_\hbar + 2\,d\,\hbar\,C_1^2\right)^{\frac{1}{2}} \\ &\leq \left(1 + \sqrt{2}(C_1 + C_2)\right)\mathcal{E}_\hbar + \sqrt{2\,d\hbar}\,C_1 \sqrt{\mathcal{E}_\hbar}, \end{aligned}$$

which leads to

$$\frac{\mathrm{d}\sqrt{\mathcal{E}_\hbar}}{\mathrm{d}t} \leq (1 + C_1 + C_2)\sqrt{\mathcal{E}_\hbar} + \sqrt{d\hbar}\,C_1 \hbar.$$

By Grönwall's inequality, it leads to

$$W_{2,\hbar}(f, \boldsymbol{\rho}_\hbar) \leq \sqrt{\mathcal{E}_\hbar} \leq \sqrt{\mathcal{E}_\hbar}(0) e^{Ct} + C_1 \sqrt{d\hbar}\,\frac{e^{Ct} - 1}{C}.$$

Minimizing the right hand side as $\boldsymbol{\gamma}_\hbar^{\mathrm{in}}$ runs through $\mathcal{C}(f^{\mathrm{in}}, \boldsymbol{\gamma}_\hbar^{\mathrm{in}})$ gives the expected result. □

When $\nabla K \in B^1_{1,\infty}$, which includes the Coulomb potential, previous proposition becomes

Proposition 1.18. *Assume*

$$\nabla K \in B^1_{1,\infty}.$$

Let $\boldsymbol{\rho}_\hbar \in \mathscr{P}$ be a solution of (Hartree) *equation and f be a solution of the* (Vlasov) *equation such that the respective spatial densities verify*

$$\rho_\hbar \in L^\infty([0,T], L^\infty)$$
$$\rho \in L^\infty([0,T], L^\infty),$$

uniformly with respect to $\hbar$. Then, for all $t \in [0,T]$, we have

$$W_{2,\hbar}(f(t), \boldsymbol{\rho}_\hbar(t)) \leq \max\left(\sqrt{d\hbar},\, W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}_\hbar^{\text{in}})^{e^{t/\sqrt{2}}} e^{\lambda(e^{t/\sqrt{2}}-1)}\right),$$

where

$$\lambda = C\left(1 + \|\nabla K\|_{B^1_{1,\infty}} \sup_{[0,T]}(\|\rho\|_{L^\infty} + \|\rho_\hbar\|_{L^\infty})\right).$$

Proof. The proof is similar to the proof of Proposition 1.17. With the same notations, we arrive at

$$\frac{\mathrm{d}\mathcal{E}_\hbar}{\mathrm{d}t} \leq \mathcal{E}_\hbar + \sqrt{2\mathcal{E}_\hbar}\,(I_1 + I_2)^{1/2}. \tag{1.72}$$

Then by corollary 1.16, we obtain

$$I_1 \leq \|\nabla K\|^2_{B^1_{1,\infty}} \max(\|\rho\|_{L^\infty}, \|\rho_\hbar\|_{L^\infty}) W_2(\rho,\rho_\hbar)^2\ \|\rho\|_{L^\infty}.$$

As in the proof of Proposition 1.17, it leads to

$$I_1 \leq C_1^2(\mathcal{E}_\hbar + d\hbar),$$

where $C_1 = \|\nabla K\|_{B^1_{1,\infty}} \max(\|\rho\|_{L^\infty}, \|\rho_\hbar\|_{L^\infty})^{1/2}\|\rho\|^{1/2}_{L^\infty}$. In order to control I_2, we use the fact that since $\nabla K \in B^1_{1,\infty}$, then, as proved in Appendix A.1 (inequality (A.5) of Proposition A.16), we have

$$\|E_\hbar\|_{B^1_{\infty,\infty}} = \|\nabla K * \rho_\hbar\|_{B^1_{\infty,\infty}} \leq \|\rho_\hbar\|_{L^\infty}\|\nabla K\|_{B^1_{1,\infty}}. \tag{1.73}$$

Then we use a result proved for example in [14, Chapter 2] which states that any function in $B^1_{\infty,\infty}$ is log-Lipschitz in the sense that for any $|x-y| < 1$, we have

$$|E_\hbar(x) - E_\hbar(y)| \leq \|E_\hbar\|_{B^1_{\infty,\infty}}|x-y|\,(1 + |\ln(|x-y|)|).$$

But for any $r \in (0,1)$, since $B^1_{1,\infty} \subsetneq L^\infty$, for any $|x-y| \geq r$, we get

$$|E_\hbar(x) - E_\hbar(y)| \leq 2\|E_\hbar\|_{L^\infty} \leq C\|E_\hbar\|_{B^1_{\infty,\infty}}\frac{|x-y|}{r}.$$

Let introduce the kernel of $\boldsymbol{\gamma}_\hbar$, $\gamma(y_1, y_2, z)$ (which still depends on t and $\hbar$) and its diagonal

$$\gamma(y, z) := \gamma(y, y, z).$$

Then, we have

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^d} |E_\hbar(y) - E_\hbar(x)|^2 \gamma(y, z) \,\mathrm{d}y \,\mathrm{d}z. \\
&\leq C_2^2 \left(\mathcal{E}_\hbar + \int_{\mathbb{R}^{2d}} \int_{|x-y|<r} |y - x|^2 \ln(|x - y|)^2 \gamma(y, z) \,\mathrm{d}y \,\mathrm{d}z \right) \\
&\leq C_2^2 \left(\mathcal{E}_\hbar + \frac{1}{4} \int_{\mathbb{R}^{2d}} \int_{|x-y|<r} F(|y - x|^2) \gamma(y, z) \,\mathrm{d}y \,\mathrm{d}z \right),
\end{aligned}$$

where $C_2 = \left(\frac{C}{r} + 1\right)^{1/2} \|\rho_\hbar\|_{L^\infty} \|\nabla K\|_{B^1_{1,\infty}}$ and $F(x) = x \ln(x)^2$. As noticed in [157], F is concave on $[0, e^{-1}]$. Thus, by taking $r = e^{-1}$, by Jensen's inequality,

$$I_2 \leq C_2^2 \left(\mathcal{E}_\hbar + \frac{1}{4} F(\mathcal{E}_\hbar) \right).$$

By combining this estimate with (1.73), equation (1.72) becomes

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{E}_\hbar}{\mathrm{d}t} &\leq \mathcal{E}_\hbar + \sqrt{2\mathcal{E}_\hbar}((C_1^2 + C_2^2)\mathcal{E}_\hbar + C_1^2 d\hbar + F(\mathcal{E}_\hbar)/4)^{1/2} \\
&\leq (1 + \sqrt{2}\,(C_1 + C_2))\mathcal{E}_\hbar + C_1 \sqrt{2d\hbar\mathcal{E}_\hbar} + \mathcal{E}_\hbar \ln(\mathcal{E}_\hbar)/\sqrt{2} \\
&\leq \lambda \mathcal{E}_\hbar + C_1 \sqrt{d\hbar/2} + \mathcal{E}_\hbar \ln(\mathcal{E}_\hbar)/\sqrt{2},
\end{aligned}$$

where $\lambda = 1 + \sqrt{2}\,(2C_1 + C_2)$ and we used the inequalities $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ and $\sqrt{2ab} \leq a + b$. Then, for any t such that $\lambda\mathcal{E}_\hbar \geq C_1\sqrt{d\hbar/2}$, we get

$$\frac{\mathrm{d}\ln(\mathcal{E}_\hbar)}{\mathrm{d}t} \leq 2\lambda + \ln(\mathcal{E}_\hbar)/\sqrt{2}.$$

By Grönwall's inequality, it leads to

$$W_{2,\hbar}(f, \boldsymbol{\rho}_\hbar) \leq \mathcal{E}_\hbar \leq \max\left(\frac{C_1(t)\sqrt{d\hbar}}{\lambda(t)\sqrt{2}},\, \mathcal{E}_\hbar(0)^{e^{t/\sqrt{2}}} e^{\sqrt{2}\tilde{\lambda}(t)(e^{t/\sqrt{2}}-1)} \right),$$

where $\tilde{\lambda}(t) = \sup_{[0,T]} \lambda$, and which gives the expected result since $C_1(t) \leq \lambda(t)$. □

Combining the propagation of moments of Theorem 1.1 with the Proposition 1.17 which gives the semiclassical convergence as soon as ρ is sufficiently integrable, we can now prove Theorem 1.3. Theorem 1.4 is proved in the same way using Proposition 1.18 and Proposition 1.13.

Proof of Theorem 1.3. Since $\nabla K \in L^\infty + L^{\mathfrak{b},\infty}$, from Theorem 1.1, we obtain the existence of $T \in (0, +\infty]$ and $\Phi \in C^0([0, T))$ such that for any $t \in [0, T)$

$$M_{n_1} < \Phi(t).$$

Moreover, from Proposition 1.6 we know that

$$\|\boldsymbol{\rho}_\hbar\|_{\mathcal{L}^r} = \|\boldsymbol{\rho}_\hbar^{\mathrm{in}}\|_{\mathcal{L}^r} \leq C.$$

By inequality (1.21), we deduce that

$$\|\rho_\hbar\|_{L^{p_{n_1}}} \leq \Phi(t)^{1-\theta}.$$

Moreover, by Proposition 1.6, we also deduce the propagation of the mass

$$\int_{\mathbb{R}^d} \rho_\hbar = \mathrm{Tr}(\boldsymbol{\rho}_\hbar) = \|\boldsymbol{\rho}_\hbar\|_{\mathcal{L}^1} = \|\boldsymbol{\rho}_\hbar^{\mathrm{in}}\|_{\mathcal{L}^1} = 1.$$

Remarking that

$$q \geq p'_{n_1} \Leftrightarrow q \geq r' + \frac{d}{n_1} \Leftrightarrow n_1 \geq \frac{d}{q-r'},$$

we get that $p := q' \in [1, p'_{n_1}]$. Moreover, since $q' \geq 2$, it also implies that $q'/2 \in [1, p'_{n_1}]$. By Hölder's inequality, it implies that for a given $\varepsilon < 1-\theta$,

$$\|\rho_\hbar\|_{L^{q'}} \leq \Phi(t)^{\varepsilon}$$
$$\|\rho_\hbar\|_{L^{q'/2}} \leq \Phi(t)^{2\varepsilon},$$

and we can use Proposition 1.17 to get the result. □

1.7 Superposition of coherent states

We recall in this section some results about the approximation of measures on the phase space by a superposition of coherent states and state some applications in our case. See also Thirring [204], Lions and Paul [152], Golse et al [104]. Let $\varphi \in L^1$ be a smooth function such that $\|\varphi\|_{L^2} = 1$. Then the coherent states are defined by

$$\varphi_{x,\xi}(y) = \frac{1}{h^{d/4}} \varphi\left(\frac{y-x}{\sqrt{h}}\right) e^{2i\pi y\cdot\xi/h},$$

and we will denote the associated density operator by

$$\boldsymbol{\rho}_{x,\xi} := |\varphi_{x,\xi}\rangle\langle\varphi_{x,\xi}|.$$

We can then associate to a measure $\mu \in \mathcal{P}(\mathbb{R}^{2d})$ of the phase space the following operator

$$\boldsymbol{\mu} := \mathrm{op}_\varphi(\mu) := \iint_{\mathbb{R}^{2d}} \boldsymbol{\rho}_{x,\xi}\, \mu(\mathrm{d}x\,\mathrm{d}\xi).$$

It corresponds to the density operator defined in [152, Exemple III.7]. Up to a constant depending on $\hbar$, this is also what is called a Töplitz operator in [104]. The constant comes from the fact that we consider operators associated to measures with finite mass on the semiclassical limit, while Töplitz operators describe operators acting on these measures.

As expected, the mass is the trace of the operator

$$\iint_{\mathbb{R}^{2d}} \mu = \mathrm{Tr}(\boldsymbol{\mu})\,.$$

Moreover, we remark that $\boldsymbol{\rho}_{x,\xi} = \mathrm{op}_\varphi(\delta_{x,\xi})$ and as proved in [152], by defining the Wigner transform $\delta^\varphi_{x,\xi} := w_\hbar(\boldsymbol{\rho}_{x,\xi})$, the following holds

$$\delta^\varphi_{x,\xi} \underset{h\to 0}{\rightharpoonup} \delta_{x,\xi}$$
$$w_\hbar(\boldsymbol{\mu}) = \delta^\varphi_{0,0} * \mu \underset{h\to 0}{\rightharpoonup} \mu, \tag{1.74}$$

where the convergence holds in the sense of the duality with $C_0(\mathbb{R}^{2n})$. An other result proved in [105] is the comparison between the Wasserstein pseudo-distance defined in (1.4) with the classical Wasserstein pseudo-distance, which completes Theorem 1.1

Proposition 1.19 (Golse and Paul [105])**.** *Let $(\mu,\nu) \in \mathcal{P}(\mathbb{R}^{2d})^2$ be two probability measures such that $W_2(\nu,\mu) < \infty$ and $\boldsymbol{\mu} := \mathrm{op}_\varphi(\mu)$ where φ is a Gaussian with $\|\varphi\|_{L^2} = 1$. Then*

$$|W_{2,\hbar}(\nu,\boldsymbol{\mu}) - W_2(\nu,\mu)| \leq \sqrt{2d\hbar}.$$

Finally, the following proposition justifies our definition (1.3) for the quantum Lebesgue norm.

Proposition 1.20. *Let $\mu \in \mathcal{P}(\mathbb{R}^{2d})$ and $\boldsymbol{\mu} := \mathrm{op}_\varphi(\mu)$. Then for any $r \geq 1$, it holds*

$$\|\mu\|_{L^r_{x,\xi}} \leq \|\boldsymbol{\mu}\|_{\mathcal{L}^r} \tag{1.75}$$
$$\|\mu\|_{L^\infty_{x,\xi}} = \|\boldsymbol{\mu}\|_{\mathcal{L}^\infty}. \tag{1.76}$$

Moreover, in the particular case $\delta^\varphi_{0,0} \geq 0$, we have for any $r \geq 2$

$$\|w_\hbar(\boldsymbol{\mu})\|_{L^r_{x,\xi}} \leq \|\mu\|_{L^r_{x,\xi}} = \|\boldsymbol{\mu}\|_{\mathcal{L}^r}, \tag{1.77}$$

with equality in the first inequality if $r = 2$, as well as the following convergences

$$\|w_\hbar(\boldsymbol{\mu})\|_{L^r_{x,\xi}} \underset{h\to 0}{\to} \|\boldsymbol{\mu}\|_{\mathcal{L}^r} \tag{1.78}$$
$$w_\hbar(\boldsymbol{\mu}) \underset{h\to 0}{\to} \mu \text{ in } L^r. \tag{1.79}$$

Remark 1.14. *The assumption $\delta^\varphi_{0,0} \geq 0$ is verified for example when $\varphi(x) = e^{-\pi|x|^2/2}$, since we can then compute explicitly*

$$\delta^\varphi_{x_0,\xi_0} = \frac{1}{h^d} e^{-\frac{\pi}{h}(|x-x_0|^2+|y-y_0|^2)}.$$

Proof. As proved in [204] or [152, Exemple III.7], for any convex mapping $F \geq 0$ such that $F(0) = 0$, it holds

$$\iint_{\mathbb{R}^{2d}} F(\mu) \frac{\mathrm{d}x\,\mathrm{d}\xi}{h^d} \leq \mathrm{Tr}\Big(F\left(\frac{\boldsymbol{\mu}}{h^d}\right)\Big).$$

By taking $F(x) = |x|^r$ for $r \geq 1$, it implies in particular

$$\|\mu\|^r_{L^r_{x,\xi}} \leq h^{-d(r-1)} \|\boldsymbol{\mu}\|^r_r = \|\boldsymbol{\mu}\|^r_{\mathcal{L}^r},$$

which proves (1.75). As noticed in [104, Appendix B], this inequality also holds in the other direction when $r = \infty$, which leads to (1.76). Then, as noticed in [152], we deduce from (1.74) that if $\delta^{\varphi}_{0,0} \geq 0$, we have

$$\iint_{\mathbb{R}^{2d}} F(w_\hbar(\boldsymbol{\mu})) \leq \iint_{\mathbb{R}^{2d}} F(\mu).$$

Taking again $F(x) = |x|^r$ leads to the first part of (1.77)

$$\|w_\hbar(\boldsymbol{\mu})\|_{L^r_{x,\xi}} \leq \|\mu\|_{L^r_{x,\xi}} \leq \|\boldsymbol{\mu}\|_{\mathcal{L}^r}.$$

However, for $r = 2$, the following equality holds for any operator $\boldsymbol{\mu}$

$$\|w_\hbar(\boldsymbol{\mu})\|_{L^2_{x,\xi}} = \|\boldsymbol{\mu}\|_{\mathcal{L}^2}.$$

Thus, the above inequalities are identities when $r = 2$

$$\|w_\hbar(\boldsymbol{\mu})\|_{L^2_{x,\xi}} = \|\mu\|_{L^2_{x,\xi}} = \|\boldsymbol{\mu}\|_{\mathcal{L}^2}.$$

By complex interpolation, we deduce from the above equation and formula (1.76) that for any $r \geq 2$, $\mathrm{op}_\varphi \in \mathscr{B}(L^r_{x,\xi}, \mathcal{L}^r)$ and

$$\|\boldsymbol{\mu}\|_{\mathcal{L}^r} \leq \|\mu\|_{L^r_{x,\xi}},$$

which proves the equality in formula (1.77). Finally, from (1.74) and (1.77), we deduce that $w_\hbar(\boldsymbol{\mu}) \rightharpoonup \mu$ in L^r and

$$\|\mu\|_{L^r_{x,\xi}} \leq \liminf_{h\to 0} \|w_\hbar(\boldsymbol{\mu})\|_{L^r_{x,\xi}},$$

which combined with (1.77) leads to (1.78) and then (1.79). □

Combining all these results, we can for example write a simplified version of Theorem 1.3 for superposition of coherent states.

Theorem 1.21. *Assume K verifies (1.11) and (1.12) and let f be a solution of the (Vlasov) equation and $\boldsymbol{\rho}_\hbar$ be a solution of (Hartree) equation with respective initial conditions*

$$f^{\mathrm{in}} \in \mathcal{P} \cap L^\infty_{x,\xi} \text{ verifying (1.6) and (1.7)}$$
$$\boldsymbol{\rho}^{\mathrm{in}}_\hbar = \mathrm{op}_\varphi(g^{\mathrm{in}}) \text{ with } g^{\mathrm{in}} \in \mathcal{P} \cap L^\infty_{x,\xi},$$

where φ is a normalized Gaussian. Assume also that the initial quantum velocity moment

$$M_{n_1}^{\mathrm{in}} < C \text{ for a given } n_1 \geq \frac{d}{q-1}. \tag{1.80}$$

Then there exists $T > 0$ such that for any $t \in (0,T)$,

$$W_2(f(t), \tilde{f}_\hbar(t)) \leq C_T \left(W_2(f^{\mathrm{in}}, g^{\mathrm{in}}) + \sqrt{\hbar} \right),$$

where $\tilde{f}_\hbar$ is the Husimi transform of $\boldsymbol{\rho}_\hbar$.

The advantage is that in the above results, the semiclassical estimate is stated only in terms of the classical Wasserstein distance which is a true distance, and it also allows to take $f^{\mathrm{in}} = g^{\mathrm{in}}$. We can do the same for Theorem 1.4. We state it here for the Coulomb potential in dimension $d = 3$.

Theorem 1.22. *Assume $K = \frac{1}{|x|}$ and let f be a solution of the (Vlasov) equation and $\boldsymbol{\rho}_\hbar$ be a solution of (Hartree) equation with respective initial conditions*

$$f^{\mathrm{in}} \in \mathcal{P} \cap L^\infty_{x,\xi} \text{ verifying (1.6) and (1.7)}$$
$$\boldsymbol{\rho}_\hbar^{\mathrm{in}} = \mathrm{op}_\varphi(g^{\mathrm{in}}) \text{ with } g^{\mathrm{in}} \in \mathcal{P} \cap L^\infty_{x,\xi},$$

where φ is a normalized Gaussian. Moreover, assume that

$$\forall \mathrm{i} \in [\![1,3]\!],\ \mathsf{p}_\mathrm{i}^4 \boldsymbol{\rho}_\hbar^{\mathrm{in}} \in \mathcal{L}^\infty,$$

where $\mathsf{p}_\mathrm{i} := -i\hbar\partial_\mathrm{i}$. Assume also that the initial quantum velocity moment

$$M_{16}^{\mathrm{in}} < C. \tag{1.81}$$

Then there exists $T > 0$ such that

$$\rho_\hbar \in L^\infty([0,T], L^\infty),$$

uniformly in $\hbar$, and there exists a constant C_T depending only on the initial conditions and independent of $\hbar$ such that

$$W_2(f(t), \tilde{f}_\hbar(t)) \leq C_T \left(W_2(f^{\mathrm{in}}, g^{\mathrm{in}}) + \sqrt{\hbar} \right).$$

Chapter 2

Global Semiclassical Limit from Hartree to Vlasov Equation for Concentrated Initial Data

Abstract

We prove a quantitative and **global in time** semiclassical limit from the Hartree to the Vlasov equation in the case of a singular interaction potential in dimension $d \geq 3$, including the case of a Coulomb singularity in dimension $d = 3$. This result holds for initial data concentrated enough in the sense that some space moments are initially sufficiently small.
As an intermediate result, we also obtain quantitative semiclassical bounds on the space and velocity moments of even order and the asymptotic behaviour of the spatial density due to dispersion effects.

Résumé

On démontre une version quantitative et **globale en temps** de la limite semiclassique de l'équation de Hartree vers l'équation de Vlasov dans les cas de potentiels singuliers en dimension $d \geq 3$, incluant le cas d'une singularité Coulombienne en dimension $d = 3$. Ce résultat nécessite que les données initiales soient suffisamment concentrées dans le sens que certains moments en espaces doivent être initialement suffisamment petits.
On obtient comme résultat intermédiaire des estimées semi-classiques sur les moments en vitesse et en espace d'ordre pair et le comportement asymptotique de la densité spatiale dû aux effets de dispersion.

Contents

2.1 Introduction

The equation governing the dynamics of a large number of interacting particles of density $f = f(t, x, \xi)$ in the phase space is the Vlasov equation

$$\partial_t f = -\xi \cdot \nabla_x f - E \cdot \nabla_\xi f, \tag{Vlasov}$$

where $E = -\nabla V$ is the force field corresponding to the mean field potential

$$V(x) = (K * \rho_f)(x) = \int_{\mathbb{R}^d} K(x-y)\rho_f(y)\,\mathrm{d}y,$$

where we denote by $\rho_f(x) := \int_{\mathbb{R}^d} f(x,v)\,\mathrm{d}v$ the spatial density. Its counterpart in quantum mechanics is the following Hartree equation

$$i\hbar\,\partial_t \boldsymbol{\rho} = [H, \boldsymbol{\rho}], \tag{Hartree}$$

where $\boldsymbol{\rho}$ is a self-adjoint Hilbert-Schmidt operator called the density operator and the Hamiltonian is defined by

$$H = -\frac{h^2}{2}\Delta + V.$$

In this formula, the potential is defined by $V = K * \rho$ where the spatial density ρ is defined as the diagonal of the kernel $\varrho(x, y)$ of the operator $\boldsymbol{\rho}$, i.e. $\rho(x) = \varrho(x, x)$.

In this chapter, we study in a quantitative way the limit when $\hbar \to 0$ of (Hartree) equation which is known to converge to the (Vlasov) equation. The question of the derivation of this equation from the quantum mechanics is a very active topic of research. Non-constructive results in weak topologies have indeed already been proved, including the case of Coulomb interactions, starting from the work of Lions and Paul [152] and Markowich and Mauser [158]. See also [110, 97, 109, 5, 4].

Some more precise quantitative results have also more recently been proved for smooth forces which are always at least Lipschitz in [11, 6, 7, 25, 104]. In [105], Golse and

Paul introduce a pseudo-distance on the model of the Wasserstein-(Monge-Kantorovitch) between classical phase space densities and quantum density operators to get a rate of convergence for the semiclassical limit for Lipschitz forces. This strategy have been used in the recent paper [133] of the present author to extend this result to more singular interactions, but only up to a fixed time in the case of potentials with a strong singularity such as the Coulomb interaction.

We also mention the work of Porta et al [183] and Saffirio [186] about the question of the mean-field limit for the Schrödinger equation to the Hartree equation for Fermions since this limit is coupled with a semiclassical limit. Results are proved for the Coulomb interaction under assumptions of propagation of regularity along the Hartree dynamics which is still an open problem. Other results about the mean-field limit can be found in [20, 89, 19] where non-quantitative results are established for the Coulomb potential, and more precise limits can be found in [185, 182, 104, 171, 105, 107, 108] for Bosons and in [96, 95, 26, 24, 13, 181, 183, 180] for Fermions.

Here, we extends the results of [133] by proving a global in time semiclassical limit under a smallness condition of space moments. We first prove a global in time bound on some modified space moments, from which we obtain the propagation of space and velocity moments. The bound on the velocity is then sufficient to use the theory already used in the above mentioned paper to get a global L^∞ bound on the spatial density and the quantitative semiclassical limit in the quantum Wasserstein pseudo-distance.

The fact that the time decay due to the dispersion properties gives global estimates for the Vlasov equation was already used in [18]. The modified space moments of order 2 are linked to a Lyapunov functional reminiscent of the conservation of energy, see [179, 82]. The propagation of modified space moments was investigated further in [63, 176, 177].

2.1.1 Main results

We first define the quantum version of the phase space Lebesgue and weighted Lebesgue spaces as

$$\begin{aligned} \mathcal{L}^p &:= \left\{\boldsymbol{\rho} \in \mathscr{B}(L^2), \|\boldsymbol{\rho}\|_{\mathcal{L}^p} := h^{-d/p'} \operatorname{Tr}(|\boldsymbol{\rho}|^p)^{\frac{1}{p}} < \infty\right\}, \\ \mathcal{L}^p_+ &:= \{\boldsymbol{\rho} \in \mathcal{L}^p, \boldsymbol{\rho} = \boldsymbol{\rho}^* \geq 0\} \\ \mathcal{L}^p(m) &:= \{\boldsymbol{\rho} \in \mathcal{L}^p, \boldsymbol{\rho} m \in \mathcal{L}^p\}. \end{aligned}$$

We also define the quantum probability measures by

$$\mathscr{P} := \left\{\boldsymbol{\rho} \in \mathcal{L}^1_+, \operatorname{Tr}(\boldsymbol{\rho}) = 1\right\}.$$

We will denote by $\mathsf{p} := -i\hbar\nabla$ the quantum impulsion, which is an unbounded operator on L^2.

Our first result states that if the spatial density is concentrated enough, then some kinetic moments are bounded globally in time.

Theorem 2.1. *Let $d \geq 3$, $n \in 2\mathbb{N}$, $r \in [1,\infty]$ and define $\mathfrak{b}_n := \frac{nr'+d}{n+1}$. Assume*

$$\nabla K \in L^{\mathfrak{b},\infty} \text{ with } \mathfrak{b} \in \left(\max\left(\tfrac{d}{3}, \mathfrak{b}_4, \mathfrak{b}_n\right), \tfrac{d}{2}\right), \tag{2.1}$$

and let $\boldsymbol{\rho}$ be a solution of (Hartree) *equation with initial condition*

$$\boldsymbol{\rho}^{\mathrm{in}} \in \mathcal{L}^r \cap \mathcal{L}^1_+(1+|x|^n+|\mathsf{p}|^n).$$

Then there exists an explicit constant $\mathcal{C} = \mathcal{C}(M_0, \mathrm{Tr}(|\mathsf{p}|^n\boldsymbol{\rho}^{\mathrm{in}}), \|\nabla K\|_{L^{\mathfrak{b},\infty}}, \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r}) > 0$ such that if

$$\mathrm{Tr}\Big(|x|^n\boldsymbol{\rho}^{\mathrm{in}}\Big) < \mathcal{C}, \tag{2.2}$$

then

$$\mathrm{Tr}(|x - t\mathsf{p}|^n\boldsymbol{\rho}) \in L^\infty(\mathbb{R}_+).$$

Remark 2.1. *The theorem applies in particular in the case of interaction kernels K with a singularity like the Coulomb interaction. For example for any $\varepsilon > 0$*

$$K(x) = \frac{\pm 1}{|x|}\mathbb{1}_{|x|\leq 1} + \frac{\pm 1}{|x|^{1+\varepsilon}}\mathbb{1}_{|x|>1}. \tag{2.3}$$

An interesting case of application is the case of the Yukawa potential that is commonly used approximation in the case when there are particles with positive and negative charge, and which is of the form

$$K(x) = \frac{e^{-|x|/\lambda_D}}{|x|},$$

where $\lambda_D > 0$ is the Debye length, which represents the characteristic size of the interaction.

Remark 2.2. *An other good example of potentials verifying the assumptions of the theorem are potentials of the form*

$$K(x) = \frac{\pm 1}{|x|^a}, \tag{2.4}$$

with $a = \frac{d}{\mathfrak{b}} - 1 \in (1, \frac{8}{7})$. In dimension $d = 4$, $d = 5$ and $d \geq 6$, one can even better take respectively $a \in (1, \frac{3}{2})$, $a \in (1, \frac{16}{9})$ and $a \in (1, 2)$

Remark 2.3. *Since $\boldsymbol{\rho} \in \mathcal{L}^1_+$, it is an Hilbert-Schmidt operator that can be written as a integral operator of kernel $\varrho(x,y)$ and it can also be diagonalized by the spectral theorem. Hence, we can write for any $\varphi \in L^2$*

$$\boldsymbol{\rho}\,\varphi(x) = \int_{\mathbb{R}^d} \varrho(x,y)\varphi(y)\,\mathrm{d}y = \sum_{j\in J} \lambda_j |\psi_j\rangle\langle\psi_j|\varphi\rangle,$$

where $(\psi_j)_{j\in J} \in (L^2)^J$ with $J \subset \mathbb{N}$ is an orthogonal basis. The space density can then be written

$$\rho(x) := \varrho(x,x) = \sum_{j\in J} \lambda_j |\varphi_j(x)|^2,$$

and the space moments

$$\mathrm{Tr}(|x|^n\,\boldsymbol{\rho}) = \int_{\mathbb{R}^d} \rho(x)\,|x|^n\,\mathrm{d}x.$$

We can state the analogue of this theorem for solutions of the (Vlasov) equation

Proposition 2.2. *Let $d \geq 3$, $n \in 2\mathbb{N}$, $r \in [1, \infty]$ and assume ∇K verifies condition (2.1). Let f be a solution of (Vlasov) equation with nonnegative initial condition*

$$f^{\mathrm{in}} \in L^r_{x,\xi} \cap L^1(1 + |x|^n + |\xi|^n).$$

Then there exists an explicit constant $\mathcal{C} > 0$ such that if

$$\iint_{\mathbb{R}^{2d}} f^{\mathrm{in}}(x,\xi)|x|^n \,\mathrm{d}x\,\mathrm{d}\xi < \mathcal{C},$$

then

$$\iint_{\mathbb{R}^{2d}} f(\cdot, x, \xi)|x - t\xi|^n \,\mathrm{d}x\,\mathrm{d}\xi \in L^\infty(\mathbb{R}_+).$$

We can use the first theorem to obtain good estimates on the space and velocity moments and on the spatial density that do not depend on $\hbar$.

Theorem 2.3. *Let $d \geq 3$, $r \in [d', \infty]$, $n \in (2\mathbb{N})\backslash\{0, 2\}$ and assume*

$$\nabla K \in L^{\mathfrak{b},\infty} \text{ with } \mathfrak{b} \in \left(\max\left(\mathfrak{b}_4, \tfrac{d}{3}\right), \tfrac{d}{2}\right), \tag{2.5}$$

and let $\boldsymbol{\rho}$ be a solution of (Hartree) equation with initial condition

$$\boldsymbol{\rho}^{\mathrm{in}} \in \mathcal{L}^r \cap \mathcal{L}^1_+(1 + |x|^4 + |\mathsf{p}|^n),$$

for a given even integer $n \geq 4$. Then there exists $\mathcal{C} = \mathcal{C}(M_0, M_4^{\mathrm{in}}, \|\nabla K\|_{L^{\mathfrak{b},\infty}}, \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r})$ such that if

$$\mathrm{Tr}\Big(|x|^4 \boldsymbol{\rho}^{\mathrm{in}}\Big) \leq \mathcal{C},$$

then there exists $c_n = c_{d,n,r} > 0$ and $C > 0$ depending on the initial conditions such that

$$\mathrm{Tr}(|\mathsf{p}|^n \boldsymbol{\rho}) \leq C\,(1 + t^{c_n}) \tag{2.6}$$

$$\mathrm{Tr}(|x|^n \boldsymbol{\rho}) \leq C\left(\mathrm{Tr}\Big(|x|^n \boldsymbol{\rho}^{\mathrm{in}}\Big) + 1 + t^{n(c_n+1)}\right) \tag{2.7}$$

$$\|\rho\|_{L^p} \leq C\, t^{-d/p'}, \tag{2.8}$$

where $p' = r' + \frac{d}{4}$.

Remark 2.4. *The constant c_4 can be taken arbitrarily close to 0.*

Again, we can state the analogue result for the Vlasov equation.

Proposition 2.4. *Let $d \geq 3$, $r \in [d', \infty]$, $n \in (2\mathbb{N})\backslash\{0, 2\}$ and assume K verifies (2.5). Let f be a solution of (Vlasov) equation with nonnegative initial condition*

$$f^{\mathrm{in}} \in L^r_{x,\xi} \cap L^1_{x,\xi}(1 + |x|^4 + |\mathsf{p}|^n),$$

for a given even integer $n \geq 4$. Then there exists $\mathcal{C} > 0$ such that if

$$\iint_{\mathbb{R}^{2d}} f^{\mathrm{in}}(x,\xi)|x|^4 \,\mathrm{d}x\,\mathrm{d}\xi \leq \mathcal{C},$$

then there exists $c_n = c_{d,n,r} > 0$ *and* $C > 0$ *depending on the initial conditions such that*

$$\iint_{\mathbb{R}^{2d}} f(t,x,\xi)|\xi|^n \,\mathrm{d}x\,\mathrm{d}\xi \leq C(1+t^{c_n})$$
$$\iint_{\mathbb{R}^{2d}} f(t,x,\xi)|x|^n \,\mathrm{d}x\,\mathrm{d}\xi \leq C(1+t^{n(c_n+1)})$$
$$\|\rho_f\|_{L^p} \leq Ct^{-d/p'},$$

where $p' = r' + \frac{d}{4}$.

Before stating the result about the semiclassical limit, we recall the definition of the semiclassical Wasserstein-(Monge-Kantorovitch) distance introduced by Golse and Paul in [105]. We say that $\boldsymbol{\gamma} \in L^1(\mathbb{R}^{2d}, \mathscr{P})$ is a semiclassical coupling between a classical kinetic density $f \in L^1 \cap \mathcal{P}(\mathbb{R}^{2d})$ and a density operator $\boldsymbol{\rho} \in \mathscr{P}$ and we write $\boldsymbol{\gamma} \in \mathcal{C}(f, \boldsymbol{\rho})$ when

$$\mathrm{Tr}(\boldsymbol{\gamma}(z)) = f(z)$$
$$\int_{\mathbb{R}^{2d}} \boldsymbol{\gamma}(z)\,\mathrm{d}z = \boldsymbol{\rho}.$$

Then we define the semiclassical Wasserstein-(Monge-Kantorovich) pseudo-distance in the following way

$$W_{2,\hbar}(f,\boldsymbol{\rho}) := \left(\inf_{\boldsymbol{\gamma}\in\mathcal{C}(f,\boldsymbol{\rho})} \int_{\mathbb{R}^{2d}} \mathrm{Tr}(\mathbf{c}_\hbar(z)\boldsymbol{\gamma}(z))\,\mathrm{d}z\right)^{\frac{1}{2}},$$

where $\mathbf{c}_\hbar(z)\varphi(y) = (|x-y|^2 + |\xi - \mathsf{p}|^2)\,\varphi(y)$, $z = (x,\xi)$ and $\mathsf{p} = -i\hbar\nabla_y$. This is not a distance but it is comparable to the classical Wasserstein distance W_2 between the Wigner transform of the quantum density operator and the normal kinetic density, in the sense of the following Theorem

Theorem 2.5 (Golse & Paul [105]). *Let* $\boldsymbol{\rho} \in \mathscr{P}$ *and* $f \in \mathcal{P}(\mathbb{R}^{2d})$ *be such that*

$$\int_{\mathbb{R}^{2d}} f(x,\xi)\left(|x|^2 + |\xi|^2\right)\mathrm{d}x\,\mathrm{d}\xi < \infty.$$

Then one has $W_{2,\hbar}(f,\boldsymbol{\rho})^2 \geq d\hbar$ *and for the Husimi transform* $\tilde{f}_\hbar$ *of* $\boldsymbol{\rho}$*, it holds*

$$W_2(f,\tilde{f}_\hbar)^2 \leq W_{2,\hbar}(f,\boldsymbol{\rho})^2 + d\hbar.$$

See also [106] for more results about this pseudo-distance.

Our last theorem uses these results to obtain the semiclassical limit. We also recall the following theorem which will gives us our assumptions on the classical solution of the (Vlasov) equation

Theorem 2.6 (Lions & Perthame [153], Loeper [157]). *Assume* $f^{\mathrm{in}} \in L^\infty_{x,\xi}(\mathbb{R}^6)$ *verify*

$$\iint_{\mathbb{R}^6} f^{\mathrm{in}}|\xi|^{n_0}\,\mathrm{d}x\,\mathrm{d}\xi < C \text{ for a given } n_0 > 6,$$

and for all $R > 0$,

$$\sup_{(y,w)\in\mathbb{R}^6} \operatorname{ess}\{f^{\mathrm{in}}(y+t\xi, w), |x-y| \leq Rt^2, |\xi - w| \leq Rt\} \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty_x L^1_\xi).$$

Then there exists a unique solution to the (Vlasov) equation with initial condition $f_{t=0} = f^{\mathrm{in}}$. Moreover, in this case, the spatial density verifies $\rho_f \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty)$.

Theorem 2.5. *Let $d \geq 3$ and assume*

$$\nabla K \in B^1_{1,\infty} \cap L^{\mathfrak{b}} \text{ with } \mathfrak{b} \in \left(\max\left(\mathfrak{b}_4, \tfrac{d}{3}\right), \tfrac{d}{2}\right).$$

Let $\boldsymbol{\rho}$ be a solution of (Hartree) equation with initial condition $\boldsymbol{\rho}^{\mathrm{in}}$ verifying

$$\begin{aligned} \boldsymbol{\rho}^{\mathrm{in}} &\in \mathcal{L}^\infty \cap \mathcal{L}^1_+(1+|\mathsf{p}|^{n_1}) \\ \forall \mathrm{i} \in [\![1,d]\!],\ \mathsf{p}_{\mathrm{i}}^{n_0}\boldsymbol{\rho}^{\mathrm{in}} &\in \mathcal{L}^\infty, \end{aligned}$$

where $\mathsf{p}_{\mathrm{i}} := -i\hbar\partial_{\mathrm{i}}$ and $(n_0, n) \in (2\mathbb{N})^2$ is such that

$$\begin{aligned} n_0 &> d \\ n &\geq \frac{d + \mathfrak{b}(n_0 - 1)}{\mathfrak{b} - 1}. \end{aligned}$$

Let f is a solution of the (Vlasov) equation with initial condition f^{in} verifying the hypotheses of Theorem 2.6 and with the same mass as $\boldsymbol{\rho}$. Then there exists $\mathcal{C}$ depending on M_0, $\mathrm{Tr}(|\mathsf{p}|^4\boldsymbol{\rho}^{\mathrm{in}})$, $\|\nabla K\|_{L^{\mathfrak{b},\infty}}$ and $\|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^\infty}$ such that if

$$\mathrm{Tr}\Big(|x|^4\boldsymbol{\rho}^{\mathrm{in}}\Big) \leq \mathcal{C},$$

then

$$\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty). \tag{2.9}$$

Moreover, the following semiclassical estimate holds

$$W_{2,\hbar}(f(t), \boldsymbol{\rho}(t)) \leq \max\left(\sqrt{d\hbar},\, W_{2,\hbar}(f^{\mathrm{in}}, \boldsymbol{\rho}^{\mathrm{in}})^{e^{t/\sqrt{2}}} e^{\lambda(e^{t/\sqrt{2}}-1)}\right),$$

with

$$\lambda = C_d\left(1 + \frac{\|\nabla K\|_{B^1_{1,\infty}}}{1 + t^{n_0(1+c_n/\mathfrak{b}')}} \sup_{t\in\mathbb{R}_+}(\|\rho_f(t)\|_{L^\infty} + \|\rho(t)\|_{L^\infty})\right),$$

where c_n is given by (2.6).

Again, the additional assumption $\nabla K \in B^1_{1,\infty}$ is compatible with a kernel with a Coulomb singularity in dimension $d = 3$ such as the one given in Remark 2.1.

2.2 Free Transport

We want to use the time decay properties of the kinetic free transport equation which writes for $f = f(t,x,\xi)$

$$\partial_t f + \xi \cdot \nabla_x f = 0.$$

In quantum mechanics, free transport is given by the free Schrödinger equation

$$i\hbar\,\partial_t \psi = H_0 \psi,$$

where $\hbar = \frac{h}{2\pi}$ and $H_0 = -\frac{\hbar^2 \Delta}{2}$ which can be written $H_0 = \frac{|\mathsf{p}|^2}{2}$ with the notation $\mathsf{p} = -i\hbar\nabla$. The solution corresponding to the initial condition ψ^{in} can be written $\mathsf{T}_t\psi^{\text{in}}$ where the semigroup T_t is given by

$$\mathsf{T}_t\psi = e^{-itH_0/\hbar}\psi = \frac{e^{-i\pi|x|^2/(ht)}}{(iht)^{d/2}} * \psi. \tag{2.10}$$

The corresponding equation for density operators $\boldsymbol{\rho} \in \mathscr{P}$ is

$$i\hbar\,\partial_t \boldsymbol{\rho} = [H_0, \boldsymbol{\rho}], \tag{2.11}$$

whose solution is $\mathsf{S}_t\boldsymbol{\rho}^{\text{in}}$ where the semigroup S_t is defined by

$$\mathsf{S}_t\boldsymbol{\rho} := \mathsf{T}_t\boldsymbol{\rho}\mathsf{T}_{-t}. \tag{2.12}$$

As it can be easily noticed, it holds $\mathsf{T}_t^* = \mathsf{T}_t^{-1} = \mathsf{T}_{-t}$ and for any $(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \in \mathscr{P}^2$, $\mathsf{S}(\boldsymbol{\rho}\boldsymbol{\rho}_2) = \mathsf{S}(\boldsymbol{\rho})\mathsf{S}(\boldsymbol{\rho}_2)$. Moreover, a straightforward computation shows that

$$\begin{aligned} \mathsf{S}_t\mathsf{p} &= \mathsf{p} \\ \mathsf{S}_t x &= x - t\mathsf{p}. \end{aligned} \tag{2.13}$$

By the spectral theory, it implies that $\mathsf{S}_t(f(x)) = f(x - t\mathsf{p})$ for any nice function f. By analogy, we can define the operator of translation of the impulsion p by

$$\begin{aligned} \tilde{\mathsf{T}}_t\psi(x) &:= e^{\frac{-\pi|x|^2 t}{ih}}\psi(x) = G_t(x)\psi(x) \\ \tilde{\mathsf{S}}_t\boldsymbol{\rho} &:= \tilde{\mathsf{T}}_t\boldsymbol{\rho}\tilde{\mathsf{T}}_{-t}, \end{aligned} \tag{2.14}$$

which verifies the equation

$$i\hbar\,\partial_t(\tilde{\mathsf{S}}_t\boldsymbol{\rho}) = \left[\frac{-|x|^2}{2}, \tilde{\mathsf{S}}_t\boldsymbol{\rho}\right],$$

and the two following relations

$$\begin{aligned} \tilde{\mathsf{S}}_t x &= x \\ \tilde{\mathsf{S}}_t\mathsf{p} &= \mathsf{p} - tx. \end{aligned} \tag{2.15}$$

We recall the quantum kinetic interpolation inequality that was already used in [133, Theorem 6]. For $k \in 2\mathbb{N}$ we define

$$\rho_k := \sum_{j\in J} \lambda_j |\mathsf{p}^{\frac{k}{2}}\psi_j|^2 = \operatorname{diag}(\mathsf{p}^{\frac{k}{2}}\boldsymbol{\rho}\cdot\mathsf{p}^{\frac{k}{2}}),$$

and for $r \geq 1$ and $0 \leq k \leq n$, we define the exponent $p_{n,k}$ by its Hölder conjugate

$$p'_{n,k} = \left(\frac{n}{k}\right)' p'_n \text{ with } p'_n = \left(r' + \frac{d}{n}\right). \tag{2.16}$$

Then the following inequality holds

Proposition 2.6. *Let $(k,n) \in (2\mathbb{N})^2$ be such that $k \leq n$ and $\boldsymbol{\rho} \in \mathcal{L}^1(|\boldsymbol{\rho}|^n) \cap \mathcal{L}^r_+$ for a given $r \in [1,\infty]$. Then there exists $C = C_{d,r,n,k} > 0$ such that*

$$\|\rho_k\|_{L^{p_{n,k}}} \leq C \operatorname{Tr}(|\mathsf{p}|^n\boldsymbol{\rho})^{1-\theta} \|\boldsymbol{\rho}\|^{\theta}_{\mathcal{L}^r}, \tag{2.17}$$

where $\theta = \frac{r'}{p'_{n,k}}$.

Corollary 2.7. *Let $n \in 2\mathbb{N}$, $r \in [1,\infty]$, $p' := r' + \frac{d}{n}$ and $\theta = \frac{r'}{p'}$. Then*

$$\|\rho\|_{L^p} \leq \frac{1}{t^{d/p'}} \operatorname{Tr}(|x - t\mathsf{p}|^n\boldsymbol{\rho})^{1-\theta} \|\boldsymbol{\rho}\|^{\theta}_{\mathcal{L}^r}.$$

Proof of Corollary 2.7. We just remark that by formula (2.15), we get

$$\begin{aligned} t^{-n}\operatorname{Tr}(|x - t\mathsf{p}|^n\boldsymbol{\rho}) &= \operatorname{Tr}(|\mathsf{p} - x/t|^n\boldsymbol{\rho}) \\ &= \operatorname{Tr}\Big(\tilde{\mathsf{S}}_{1/t}(|\mathsf{p}|^n)\boldsymbol{\rho}\Big) = \operatorname{Tr}\Big(|\mathsf{p}|^n\tilde{\mathsf{S}}_{-1/t}(\boldsymbol{\rho})\Big)\,. \end{aligned}$$

Moreover, the following also holds

$$\begin{aligned} \operatorname{diag}(\tilde{\mathsf{S}}_t\boldsymbol{\rho}) &= G_t(x)\varrho(x,x)G_{-t}(x) = \rho(x) \\ \|\tilde{\mathsf{S}}_{-1/t}\boldsymbol{\rho}\|_{\mathcal{L}^r} &= \|\boldsymbol{\rho}\|_{\mathcal{L}^r}. \end{aligned}$$

Then by the interpolation inequality (2.17) we get

$$\begin{aligned} \|\rho\|_{L^p} &= \Big\|\operatorname{diag}(\tilde{\mathsf{S}}_{-1/t}\boldsymbol{\rho})\Big\|_{L^p} \\ &\leq \operatorname{Tr}\Big(|\mathsf{p}|^n\tilde{\mathsf{S}}_{-1/t}(\boldsymbol{\rho})\Big)^{1-\theta} \|\tilde{\mathsf{S}}_{-1/t}\boldsymbol{\rho}\|^{\theta}_{\mathcal{L}^r} \\ &\leq t^{-n(1-\theta)}\operatorname{Tr}(|x - t\mathsf{p}|^n\boldsymbol{\rho})^{1-\theta} \|\boldsymbol{\rho}\|^{\theta}_{\mathcal{L}^r}. \end{aligned}$$

Finally, we remark that $n\,(1-\theta) = \frac{n(r'+d/n-r')}{r'+d/n} = \frac{d}{p'}$ to get the result. □

2.3 Propagation of moments

2.3.1 Classical case.

In this section, we define the classical kinetic, velocity and space moments by

$$\begin{aligned} L_n &:= \iint_{\mathbb{R}^{2d}} f(t,x,\xi)|x-t\xi|^n \,\mathrm{d}x\,\mathrm{d}\xi \\ M_n &:= \iint_{\mathbb{R}^{2d}} f(t,x,\xi)|\xi|^n \,\mathrm{d}x\,\mathrm{d}\xi \\ N_n &:= \iint_{\mathbb{R}^{2d}} f(t,x,\xi)|x|^n \,\mathrm{d}x\,\mathrm{d}\xi. \end{aligned}$$

Proposition 2.8 (Classical large time estimate)**.** *Let $(r,\mathfrak{b}) \in [1,\infty]\times[\mathfrak{b}_n,\infty]$, $\nabla K \in L^{\mathfrak{b},\infty}$ and $f \in L^\infty(\mathbb{R}_+, L^r_{x,\xi}\cap L^1_{x,\xi})$ be a nonnegative solution of (Vlasov) equation. Then for any $n \in 2\mathbb{N}$, there exists a constant $C = C_{d,r,n} > 0$ such that*

$$\left|\frac{\mathrm{d}L_n}{\mathrm{d}t}\right| \leq C\|\nabla K\|_{L^{\mathfrak{b},\infty}} M_0^{\Theta_0} \|f^{\mathrm{in}}\|_{L^r_{x,\xi}}^{\frac{r'}{\mathfrak{b}}} \frac{L_n^{1+\frac{a}{n}}(t)}{t^a},$$

where $a = \frac{d}{\mathfrak{b}} - 1$ and $\Theta_0 = 1 - \frac{a}{n} - \frac{r'}{\mathfrak{b}}$.

***Proof*.** We write $f = f(t,x,\xi)$, $E = E(x)$ and we compute

$$\begin{aligned} \frac{\mathrm{d}L_n}{\mathrm{d}t} &= \iint_{\mathbb{R}^{2d}} |x-t\xi|^n(-\xi\cdot\nabla_x f - E\cdot\nabla_\xi f) - |x-t\xi|^{n-2}(x-t\xi)\cdot\xi f \,\mathrm{d}x\,\mathrm{d}\xi \\ &= -t\iint_{\mathbb{R}^{2d}} f|x-t\xi|^{n-2}(x-t\xi)\cdot E\,\mathrm{d}\xi\,\mathrm{d}x. \end{aligned}$$

By Hölder's inequality, we deduce for $t \geq 0$ and any $p \in [1,\infty]$

$$\begin{aligned} \left|\frac{\mathrm{d}L_n}{\mathrm{d}t}\right| &\leq t\left\|\int_{\mathbb{R}^d} f\,|x-t\xi|^{n-1}\,\mathrm{d}\xi\right\|_{L^p_x} \|E\|_{L^{p'}} \\ &\leq t^n\left\|\int_{\mathbb{R}^d} f\,|\tfrac{x}{t}-\xi|^{n-1}\,\mathrm{d}\xi\right\|_{L^p_x} \|\nabla K * \rho_f\|_{L^{p'}} \\ &\leq C_K t^n\left\|\int_{\mathbb{R}^d} f(t,x,\xi+\tfrac{x}{t})\,|\xi|^{n-1}\,\mathrm{d}\xi\right\|_{L^p_x} \left\|\int_{\mathbb{R}^d} f\,\mathrm{d}\xi\right\|_{L^q}, \end{aligned}$$

where we used Hardy-Littlewood-Sobolev's inequality with $C_K = \|\nabla K\|_{L^{\mathfrak{b},\infty}}$ and

$$\frac{1}{p'} + \frac{1}{q'} = \frac{1}{\mathfrak{b}}. \tag{2.18}$$

Then we want to use the classical kinetic interpolation inequality which tells that for $p'_{n,k} = \left(\frac{n}{k}\right)'\left(r' + \frac{d}{n}\right)$ and $g = g(x,\xi) \geq 0$, it holds

$$\left\|\int_{\mathbb{R}^d} g(x,\xi)|\xi|^k\,\mathrm{d}\xi\right\|_{L^{p_{n,k}}} \leq C_{d,r,n}\left(\iint_{\mathbb{R}^{2d}} g(x,\xi)|\xi|^n\,\mathrm{d}\xi\,\mathrm{d}x\right)^{1-r'/p'_{n,k}} \|g\|_{L^\infty_{x,\xi}}^{r'/p'_{n,k}}. \tag{2.19}$$

Since
$$\frac{1}{p'_{n,n-1}} + \frac{1}{p'_n} = \frac{1}{r' + d/n}\left(1 - \frac{n-1}{n} + 1\right) = \frac{n+1}{nr' + d} = \frac{1}{\mathfrak{b}_n} > \frac{1}{\mathfrak{b}},$$
we can choose $p \leq p_{n,n-1}$ and $q \leq p_{n,0}$ verifying (2.18). Take $p := p_{n,n-1}$. Then $1 < q < p_n$ and by interpolation
$$\left\| \int_{\mathbb{R}^d} f \, \mathrm{d}\xi \right\|_{L^q} \leq M_0^{1-\frac{p'_n}{q'}} \left\| \int_{\mathbb{R}^d} f \, \mathrm{d}\xi \right\|_{L^{p_n}}^{p'_n/q'}.$$
Using the above inequality and then the interpolation inequality (2.19) for $k = 0$ and $k = n-1$ yields
$$\begin{aligned} \left|\frac{\mathrm{d}L_n}{\mathrm{d}t}\right| &\leq C_{d,r,n} C_K t^n M_0^{1-\frac{p'_n}{q'}} \left\| \int_{\mathbb{R}^d} f(t, x, \xi + \tfrac{x}{t}) \, |\xi|^{n-1} \, \mathrm{d}\xi \right\|_{L^p_x} \left\| \int_{\mathbb{R}^d} f \, \mathrm{d}\xi \right\|_{L^{p_n}}^{p'_n/q'} \\ &\leq C_{d,r,n} C_K t^n M_0^{1-\frac{r'}{\mathfrak{b}}+\frac{1}{n}\left(1-\frac{d}{\mathfrak{b}}\right)} \|f\|_{L^\infty_{x,\xi}}^{\frac{r'}{\mathfrak{b}}} \left(\iint_{\mathbb{R}^{2d}} f(t, x, \xi + \tfrac{x}{t}) \, |\xi|^n \, \mathrm{d}\xi \right)^{1+\frac{1}{n}\left(\frac{d}{\mathfrak{b}}-1\right)}. \end{aligned}$$
With $a = \frac{d}{\mathfrak{b}} - 1$ and by a change of variable, we get
$$\begin{aligned} \left|\frac{\mathrm{d}L_n}{\mathrm{d}t}\right| &\leq C_{d,r,n} C_K t^n M_0^{1-\frac{r'}{\mathfrak{b}}-\frac{a}{n}} \|f\|_{L^\infty_{x,\xi}}^{\frac{r'}{\mathfrak{b}}} \left(\iint_{\mathbb{R}^{2d}} f(t, x, \xi) \, |\xi - \tfrac{x}{t}|^n \, \mathrm{d}\xi \right)^{1+\frac{a}{n}} \\ &\leq C_{d,r,n} C_K t^{-a} M_0^{1-\frac{r'}{\mathfrak{b}}-\frac{a}{n}} \|f\|_{L^\infty_{x,\xi}}^{\frac{r'}{\mathfrak{b}}} \left(\iint_{\mathbb{R}^{2d}} f(t, x, \xi) \, |x - t\xi|^n \, \mathrm{d}\xi \right)^{1+\frac{a}{n}}, \end{aligned}$$
which is the expected inequality. □

2.3.2 Boundedness of kinetic moments

We define $\tilde{\boldsymbol{\rho}} := \tilde{\mathsf{S}}_{-1/t}(\boldsymbol{\rho})$ and for $n \in 2\mathbb{N}$
$$\begin{aligned} \tilde{\rho}_n &:= \operatorname{diag}(\mathsf{p}^{n/2} \tilde{\boldsymbol{\rho}} \cdot \mathsf{p}^{n/2}) \\ l_n &:= t^n \tilde{\rho}_n. \end{aligned}$$
We also introduce the following notations for the kinetic, velocity and space moments
$$\begin{aligned} L_n &:= \operatorname{Tr}(|x - t\mathsf{p}|^n \boldsymbol{\rho}) \\ M_n &:= \operatorname{Tr}(|\mathsf{p}|^n \boldsymbol{\rho}) \\ N_n &:= \operatorname{Tr}(|x|^n \boldsymbol{\rho}), \end{aligned}$$
as well as the corresponding moments $\tilde{M}_n$ and $\tilde{N}_n$ for $\tilde{\rho}$. In particular, since we have
$$\begin{aligned} L_n = \operatorname{Tr}(|x - t\mathsf{p}|^n \boldsymbol{\rho}) &= t^n \operatorname{Tr}(|\mathsf{p} - x/t|^n \boldsymbol{\rho}) \\ &= t^n \operatorname{Tr}(|\mathsf{p}|^n \tilde{\boldsymbol{\rho}}) = t^n \operatorname{Tr}\left(\mathsf{p}^{n/2} \tilde{\boldsymbol{\rho}} \cdot \mathsf{p}^{n/2}\right), \end{aligned}$$
we obtain with these notations $L_n = \int_{\mathbb{R}^d} l_n$, $M_n = \int_{\mathbb{R}^d} \rho_n$ and $N_n = \int_{\mathbb{R}^d} \rho(x)|x|^n \, \mathrm{d}x$. The following interpolation inequalities hold.

Proposition 2.9. *Let* $0 \leq k \leq n$ *and* $p'_{n,k} := \left(\frac{n}{k}\right)' p'_n$ *and* $p \in [1, p_{n,k}]$. *Then for any* $\alpha \leq k$, *there exists a constant* $C = C_{d,r,n,k} > 0$ *such that*

$$\|\rho_k\|_{L^p} \leq C M_\alpha^{1-\theta_{n,k,\alpha}} M_n^{\theta_{n,k,\alpha} - \frac{r'}{p'}} \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^{\frac{r'}{p'}} \tag{2.20}$$

$$\|l_k\|_{L^p} \leq C t^{-d/p'} L_\alpha^{1-\theta_{n,k,\alpha}} L_n^{\theta_{n,k,\alpha} - \frac{r'}{p'}} \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^{\frac{r'}{p'}}, \tag{2.21}$$

where

$$\theta_{n,k,\alpha} = \frac{p'_{n,\alpha}}{p'} + \frac{k-\alpha}{n-\alpha}.$$

Proof. By the kinetic interpolation inequality (2.17)

$$\|\rho_k\|_{L^{p_{n,k}}} \leq C_{d,r,n,k} M_n^{1-\frac{r'}{p'_{n,k}}} \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^{\frac{r'}{p'_{n,k}}}.$$

Therefore, since $p \leq p_{n,k}$, by interpolation between L^p spaces, we get

$$\begin{aligned}\|\rho_k\|_{L^p} &\leq \|\rho_k\|_{L^1}^{1-\theta} \|\rho_k\|_{L^{p_{n,k}}}^{\theta} \\ &\leq C M_k^{1-\theta} M_n^{\theta - \frac{r'}{p'}} \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^{\frac{r'}{p'}},\end{aligned}$$

where $\theta = \theta_{n,k,k} = \frac{p'_{n,k}}{p'}$ and we used the fact that $\|\rho_k\|_{L^1} = M_k$. It already proves inequality (2.20) for $k = \alpha$. Since $k \in [\alpha, n]$, we can also bound M_k in the following way

$$M_k \leq M_\alpha^{1-\frac{k-\alpha}{n-\alpha}} M_n^{\frac{k-\alpha}{n-\alpha}},$$

which yields inequality (2.20). To get (2.21), we follow the proof of Corollary 2.7. Since $\tilde{\mathsf{S}}$ preserves the Schatten norms, we can write

$$\|\tilde{\boldsymbol{\rho}}\|_{\mathcal{L}^r} = \|\boldsymbol{\rho}\|_{\mathcal{L}^r}.$$

Hence, by replacing $\boldsymbol{\rho}$ by $\tilde{\boldsymbol{\rho}}$ in the kinetic interpolation inequality (2.17) and multiplying by t^k, we obtain

$$\begin{aligned}\|l_k\|_{L^{p_{n,k}}} = t^k \|\tilde{\rho}_k\|_{L^{p_{n,k}}} &\leq C t^k \left(\mathrm{Tr}\Big(|\mathsf{p}|^n \tilde{\mathsf{S}}_{-1/t}\boldsymbol{\rho}\Big)\right)^{1-\frac{r'}{p'_{n,k}}} \|\tilde{\mathsf{S}}_{-1/t}\boldsymbol{\rho}\|_{\mathcal{L}^r}^{\frac{r'}{p'_{n,k}}} \\ &\leq C t^k \left(\mathrm{Tr}(|\mathsf{p} - x/t|^n \boldsymbol{\rho})\right)^{1-\frac{r'}{p'_{n,k}}} \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^{\frac{r'}{p'_{n,k}}} \\ &\leq C t^{k-n+\frac{nr'}{p'_{n,k}}} L_n^{1-\frac{r'}{p'_{n,k}}} \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^{\frac{r'}{p'_{n,k}}}.\end{aligned}$$

Next we remark that

$$\begin{aligned}k - n + \frac{nr'}{p'_{n,k}} &= k - n + \left(1 - \frac{k}{n}\right) \frac{nr'}{r' + d/n} = -(n-k)\left(1 - \frac{r'}{r' + d/n}\right) \\ &= -(n-k)\left(\frac{d/n}{r' + d/n}\right) = -\frac{d}{p'_{n,k}},\end{aligned}$$

and we deduce inequality (2.21) again by interpolation of L^p between L^1 and $L^{p_{n,k}}$ and by interpolation of L_k between L_α and L_n. □

Proposition 2.10 (Large time estimate)**.** *Let* $(r, \mathfrak{b}) \in [1, \infty] \times [\mathfrak{b}_n, \infty]$*,* $\nabla K \in L^{\mathfrak{b},\infty}$ *and* $\boldsymbol{\rho} \in L^\infty(\mathbb{R}_+, \mathcal{L}^r \cap \mathcal{L}^1_+)$ *be a solution of* (Hartree) *equation. Then for any* $n \in 2\mathbb{N}$*, there exists a constant* $C = C_{d,r,n} > 0$ *such that*

$$\left|\frac{\mathrm{d}L_n}{\mathrm{d}t}\right| \leq C \|\nabla K\|_{L^{\mathfrak{b},\infty}} M_0^{\Theta_0} \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r}^{\frac{r'}{\mathfrak{b}}} \frac{L_n^{1+\frac{a}{n}}(t)}{t^a},$$

where $a = \frac{d}{\mathfrak{b}} - 1$ *and* $\Theta_0 = 1 - \frac{a}{n} - \frac{r'}{\mathfrak{b}}$*.*

Proof. We first remark that by formula (2.13) and spectral theory, we deduce $|x - t\mathsf{p}|^n = \mathsf{S}_t(|x|^n)$. Therefore by defining $H_0 := \frac{|\mathsf{p}|^2}{2}$, by definition of S_t

$$i\hbar\, \partial_t(\mathsf{S}_t(|x|^n)) = [H_0, \mathsf{S}_t(|x|^n)] = [H_0, |x - t\mathsf{p}|^n] \,.$$

Hence, by differentiating L_n with respect to time, we obtain

$$\begin{aligned} i\hbar\, \partial_t L_n &= \mathrm{Tr}([H_0, |x - t\mathsf{p}|^n]\, \boldsymbol{\rho} + |x - t\mathsf{p}|^n\, [H_0 + V, \boldsymbol{\rho}]) \\ &= \mathrm{Tr}([H_0, |x - t\mathsf{p}|^n]\, \boldsymbol{\rho} + [|x - t\mathsf{p}|^n, H_0 + V]\, \boldsymbol{\rho}) \\ &= \mathrm{Tr}([|x - t\mathsf{p}|^n, V]\, \boldsymbol{\rho}) \,. \end{aligned}$$

Then we use the operator $\tilde{\mathsf{S}}_t$ of translation in the x direction defined in (2.14). By formulas (2.15) and spectral theory, we deduce that for any $t \in \mathbb{R}$, $\tilde{\mathsf{S}}_t V = V$. Therefore, we deduce

$$\begin{aligned} i\hbar\, \partial_t L_n &= t^n\, \mathrm{Tr}([(|\mathsf{p} - x/t|^n), V]\, \boldsymbol{\rho}) \\ &= t^n\, \mathrm{Tr}\Big(\Big[\tilde{\mathsf{S}}_{1/t}(|\mathsf{p}|^n), V\Big]\, \boldsymbol{\rho}\Big) \\ &= t^n\, \mathrm{Tr}\Big(\Big[\tilde{\mathsf{S}}_{1/t}(|\mathsf{p}|^n), \tilde{\mathsf{S}}_{1/t}(V)\Big]\, \boldsymbol{\rho}\Big) \\ &= t^n\, \mathrm{Tr}\Big(\tilde{\mathsf{S}}_{1/t}([|\mathsf{p}|^n, V])\boldsymbol{\rho}\Big) \\ &= t^n\, \mathrm{Tr}([|\mathsf{p}|^n, V]\, \tilde{\boldsymbol{\rho}}) \,. \end{aligned}$$

As it has been proved in [133, Proof of Theorem 3, Step 1], this expression can be bounded in the following way

$$\frac{1}{i\hbar} \mathrm{Tr}([|\mathsf{p}|^n, V]\, \tilde{\boldsymbol{\rho}}) \leq C_K \tilde{M}_n^{\frac{1}{2}} \sup_{|a+b+c| = n/2-1} \left\|\tilde{\rho}_{2|a|}\right\|_{L^\alpha}^{\frac{1}{2}} \left\|\tilde{\rho}_{2|b|}\right\|_{L^\beta}^{\frac{1}{2}} \left\|\tilde{\rho}_{2|c|}\right\|_{L^\gamma}^{\frac{1}{2}},$$

where $(a, b, c) \in (\mathbb{N}^d)^3$ are multi-indices with $|a| = a_1 + ... + a_d$ and

$$\begin{aligned} \frac{2}{\mathfrak{b}} &= \frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'} \\ C_K &= C_{d,n} \|\nabla K\|_{L^{\mathfrak{b},\infty}}. \end{aligned} \tag{2.22}$$

As in [133, Proof of Theorem 3, Step 2], we remark that for the exponents $p_{n,k}$ defined in (2.16) and multi-indices such that $|a + b + c| = n/2 - 1$, we have

$$\frac{1}{p'_{n,2|a|}} + \frac{1}{p'_{n,2|b|}} + \frac{1}{p'_{n,2|b|}} = \frac{1}{p'_n} \left(3 - \frac{2|a| + 2|b| + 2|c|}{n}\right) = \frac{2\,(n+1)}{nr' + d} = \frac{2}{\mathfrak{b}_n}.$$

Therefore, since $\mathfrak{b} \geq \mathfrak{b}_n$, we can find $(\alpha, \beta, \gamma) \in [1, p_{n,2|a|}] \times [1, p_{n,2|b|}] \times [1, p_{n,2|b|}]$ verifying (2.22) and use the interpolation inequality (2.21) for $\alpha = 0$. By the definition of l_k and the fact that $L_0 = M_0$, we deduce

$$\begin{aligned}
\partial_t L_n &\leq C_K t^{n-n/2-(|a|+|b|+|c|)} L_n^{\frac{1}{2}} \sup_{|a+b+c|=n/2-1} \left\| l_{2|a|} \right\|_{L^\alpha}^{\frac{1}{2}} \left\| l_{2|b|} \right\|_{L^\beta}^{\frac{1}{2}} \left\| l_{2|c|} \right\|_{L^\gamma}^{\frac{1}{2}} \\
&\leq C_K t L_n^{\frac{1}{2}} \sup_{|a+b+c|=n/2-1} \left\| l_{2|a|} \right\|_{L^\alpha}^{\frac{1}{2}} \left\| l_{2|b|} \right\|_{L^\beta}^{\frac{1}{2}} \left\| l_{2|c|} \right\|_{L^\gamma}^{\frac{1}{2}} \\
&\leq (C_{d,r,n} C_K M_0^{\Theta_0} \|\boldsymbol{\rho}\|_{\mathcal{L}^r}^{\Theta_1}) t^{-a} L_n^{\Theta},
\end{aligned}$$

where

$$\begin{aligned}
a &= \frac{d}{2}\left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'}\right) - 1 = \frac{d}{\mathfrak{b}} - 1 \\
\Theta_0 &= \frac{1}{2}\left(3 - p_n'\left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'}\right) - \frac{2|a| + 2|b| + 2|c|}{n}\right) \\
&= 1 + \frac{1}{n} - \frac{p_n'}{\mathfrak{b}} = 1 - \frac{a}{n} - \frac{r'}{\mathfrak{b}} \\
\Theta_1 &= \frac{r'}{2}\left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'}\right) = \frac{r'}{\mathfrak{b}} \\
\Theta &= \frac{1}{2}\left(1 + p_n'\left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'}\right) + \frac{2|a| + 2|b| + 2|c|}{n} - r'\left(\frac{1}{\alpha'} + \frac{1}{\beta'} + \frac{1}{\gamma'}\right)\right) \\
&= 1 + \frac{1}{n}\left(\frac{d}{\mathfrak{b}} - 1\right) = 1 + \frac{a}{n}.
\end{aligned}$$

We conclude by recalling that $\|\boldsymbol{\rho}\|_{\mathcal{L}^r} = \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r}$ since the Hartree equation preserves the Schatten norm. □

To prove the short time estimate, we first need the following lemma.

Lemma 2.11. *Let $n \in \mathbb{N}$. Then for any $\varepsilon > 0$, there exists $C_\varepsilon = C_{n,\varepsilon}$ such that for any operator $\boldsymbol{\rho} \geq 0$ and any $t \geq 0$*

$$\mathrm{Tr}\Big(|x - t\mathsf{p}|^{2n}\boldsymbol{\rho}\Big) \leq (1+\varepsilon)\,\mathrm{Tr}\Big(|x|^{2n}\boldsymbol{\rho}\Big) + C_\varepsilon\,\mathrm{Tr}\Big(\Big(|t\mathsf{p}|^{2n} + |\hbar t|^n\Big)\,\boldsymbol{\rho}\Big).$$

Proof. We can assume that $n \geq 1$. We first write

$$\begin{aligned}
\mathrm{Tr}\Big(|x - t\mathsf{p}|^{2n}\boldsymbol{\rho}\Big) &= \mathrm{Tr}\Big((|x|^2 - t\mathsf{p}\cdot x - tx\cdot\mathsf{p} + t^2|\mathsf{p}|^2)^n\boldsymbol{\rho}\Big) \\
&= \mathrm{Tr}\Big(|x|^{2n}\boldsymbol{\rho}\Big) + \mathrm{Tr}\Big(|t\mathsf{p}|^{2n}\boldsymbol{\rho}\Big) \\
&\quad + \sum_{k=1}^{2n-1} \sum_{\mathrm{i}\in[\![1,d]\!]^{2n}} \sum_{a\in\mathcal{A}_k^{2n}(\mathrm{i})} C_a\,\mathrm{Tr}(a_{\mathrm{i}_1}\ldots a_{\mathrm{i}_{2n}}\boldsymbol{\rho}).
\end{aligned} \tag{2.23}$$

where

$$\mathcal{A}_k^n(\mathrm{i}) = \{a = (a_{\mathrm{i}_1}, \ldots a_{\mathrm{i}_n}), \forall j \in [\![1,n]\!], a_{\mathrm{i}_j} = x_{\mathrm{i}_j} \text{ or } a_{\mathrm{i}_j} = t\mathsf{p}_{\mathrm{i}_j}, |\{j, a_j = t\mathsf{p}_j\}| = k\}.$$

Then for any $\varepsilon > 0$ we proceed by recurrence to prove that for any $m \leq 2n$ it holds

$$\begin{aligned}&\forall k \in (1, m-1), \forall \mathrm{i} \in [\![1, d]\!]^{2n}, \forall a \in \mathcal{A}_k^m,\\ &\qquad |\hbar t|^r \left|\mathrm{Tr}(a_{\mathrm{i}_1}...a_{\mathrm{i}_m}\boldsymbol{\rho})\right| \leq \varepsilon \,\mathrm{Tr}\Big(|x|^{2n}\boldsymbol{\rho}\Big) + C_\varepsilon \,\mathrm{Tr}\Big((|t\mathsf{p}|^{2n} + |\hbar t|^n)\boldsymbol{\rho}\Big), \quad (2.24)\end{aligned}$$

where $2r = 2n - m$.

Step 1. Case $m = 2$***.*** In this case for any $\epsilon > 0$, by Hölder's and Young's inequalities, there exists $C_\epsilon > 0$ such that

$$\begin{aligned}|\mathrm{Tr}(x_{\mathrm{i}} t\mathsf{p}_j \boldsymbol{\rho})| &\leq |\hbar t|^{n-1} \,\mathrm{Tr}\Big(|x|^2\boldsymbol{\rho}\Big)^{\frac{1}{2}} \,\mathrm{Tr}\Big(|t\mathsf{p}|^2\boldsymbol{\rho}\Big)^{\frac{1}{2}}\\ &\leq \epsilon \,\mathrm{Tr}\Big(|x|^2\boldsymbol{\rho}\Big) + C_\epsilon \,\mathrm{Tr}\Big(|t\mathsf{p}|^2\boldsymbol{\rho}\Big),\end{aligned}$$

where we used the fact that $\mathrm{Tr}(|x_j|^2\boldsymbol{\rho}) \leq \mathrm{Tr}(|x|^2\boldsymbol{\rho})$ and $\mathrm{Tr}(|\mathsf{p}_j|^2\boldsymbol{\rho}) \leq \mathrm{Tr}(|\mathsf{p}|^2\boldsymbol{\rho})$. Then using again Hölder's and Young's inequalities, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}|\hbar t|^{n-1} \left|\mathrm{Tr}(x_{\mathrm{i}} t\mathsf{p}_j \boldsymbol{\rho})\right| &\leq \epsilon \,\mathrm{Tr}\Big(|x|^{2n}\boldsymbol{\rho}\Big)^{\frac{1}{n}} \,\mathrm{Tr}(|\hbar t|^n\boldsymbol{\rho})^{\frac{n-1}{n}} + C_\epsilon \,\mathrm{Tr}\Big(|t\mathsf{p}|^{2n}\boldsymbol{\rho}\Big)^{\frac{1}{n}} \,\mathrm{Tr}(|\hbar t|^n\boldsymbol{\rho})^{\frac{n-1}{n}}\\ &\leq \varepsilon \,\mathrm{Tr}\Big(|x|^{2n}\boldsymbol{\rho}\Big) + C_\varepsilon \,\mathrm{Tr}\Big((|t\mathsf{p}|^{2n} + |\hbar t|^n)\boldsymbol{\rho}\Big).\end{aligned}$$

Step 2. Case $m > 2$***.*** Since we have the following commutation relations for $j \neq k \in [\![1, d]\!]^2$

$$[x_j, t\mathsf{p}_k] = [\mathsf{p}_j, \mathsf{p}_k] = [x_j, x_k] = [x_j, x_j] = 0 \quad (2.25)$$

$$[x_j, t\mathsf{p}_j] = i\hbar t, \quad (2.26)$$

any commutation operation of a_j in $(i\hbar t)^{r_0} \,\mathrm{Tr}(a_{\mathrm{i}_1}...a_{\mathrm{i}_m}\boldsymbol{\rho})$ involving r commutations of the form (2.26) will create terms of the form

$$\pm(i\hbar t)^{r_0+r} \,\mathrm{Tr}\Big(a_{\mathrm{i}_1}...a_{\mathrm{i}_{m-2r}}\boldsymbol{\rho}\Big),$$

which will be bounded using the recurrence hypothesis, so that we can assume that all the operators commute. Let $k \in (1, m-1)$ and $a \in \mathcal{A}_k^m$. Then, by using m times Hölder's inequality and then Young's inequality, we get

$$\begin{aligned}|\mathrm{Tr}(a_{\mathrm{i}_1}...a_{\mathrm{i}_m}\boldsymbol{\rho})| &\leq \mathrm{Tr}(|a_{\mathrm{i}_1}|^m\boldsymbol{\rho})^{\frac{1}{m}} \,...\, \mathrm{Tr}(|a_{\mathrm{i}_m}|^m\boldsymbol{\rho})^{\frac{1}{m}}\\ &\leq \mathrm{Tr}(|t\mathsf{p}|^m\boldsymbol{\rho})^{\frac{k}{m}} \,\mathrm{Tr}(|x|^m\boldsymbol{\rho})^{\frac{m-k}{m}}\\ &\leq \epsilon \,\mathrm{Tr}(|t\mathsf{p}|^m\boldsymbol{\rho}) + C_\epsilon \,\mathrm{Tr}(|x|^m\boldsymbol{\rho})\end{aligned}$$

Then using again Hölder's and Young's inequalities and the fact that $\frac{r}{n} = \frac{2n-m}{2n}$, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}|\hbar t|^r \left|\mathrm{Tr}(x_{\mathrm{i}} t\mathsf{p}_j \boldsymbol{\rho})\right| &\leq \epsilon \,\mathrm{Tr}\Big(|x|^{2n}\boldsymbol{\rho}\Big)^{\frac{m}{2n}} \,\mathrm{Tr}(|\hbar t|^n\boldsymbol{\rho})^{\frac{2n-m}{2n}} + C_\epsilon \,\mathrm{Tr}\Big(|t\mathsf{p}|^{2n}\boldsymbol{\rho}\Big)^{\frac{m}{2n}} \,\mathrm{Tr}(|\hbar t|^n\boldsymbol{\rho})^{\frac{2n-m}{2n}}\\ &\leq \varepsilon \,\mathrm{Tr}\Big(|x|^{2n}\boldsymbol{\rho}\Big) + C_\varepsilon \,\mathrm{Tr}\Big((|t\mathsf{p}|^{2n} + |\hbar t|^n)\boldsymbol{\rho}\Big).\end{aligned}$$

Step 3. Conclusion. Thus, coming back to formula (2.23), we obtain for any $\varepsilon > 0$ the existence of $C_\varepsilon > 0$ such that

$$\begin{aligned}\mathrm{Tr}\Big(|x - t\mathsf{p}|^{2n}\boldsymbol{\rho}\Big) \leq \mathrm{Tr}\Big(|x|^{2n}\boldsymbol{\rho}\Big) + \mathrm{Tr}\Big(|t\mathsf{p}|^{2n}\boldsymbol{\rho}\Big) \\ + \varepsilon\,\mathrm{Tr}\Big(|x|^{2n}\boldsymbol{\rho}\Big) + C_\varepsilon\,\mathrm{Tr}\Big((|t\mathsf{p}|^{2n} + |\hbar t|^n)\boldsymbol{\rho}\Big)\,,\end{aligned}$$

which proves the result. □

To get a short time kinetic moment estimate, we use [133, Theorem 3] which tells us that for any $n \in 2\mathbb{N}$ and $\mathfrak{b} > \max(\mathfrak{b}_4, \mathfrak{b}_n)$, there exists a time

$$T = T_{\|\nabla K\|_{L^{\mathfrak{b},\infty}},\|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r},M_0,M_n^{\mathrm{in}},d,r,n}, \tag{2.27}$$

and a positive constant m depending on ∇K, $\|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r}$, M_0, M_n^{in}, d, r and n such that

$$\forall (k,t) \in [0,n] \times [0,T],\ M_k(t) \leq m. \tag{2.28}$$

Proposition 2.12 (Short time estimate)**.** *Let $n \in 2\mathbb{N}\backslash\{0\}$, $r \in [1,\infty]$,*

$$\nabla K \in L^{\mathfrak{b},\infty} \text{ for } \mathfrak{b} \in [\max(\mathfrak{b}_4, \mathfrak{b}_n), \infty],$$

and $\boldsymbol{\rho} \in L^\infty([0,T], \mathcal{L}^r \cap \mathcal{L}^1_+(1 + |x|^n + |\boldsymbol{\rho}|^n))$ be a solution of (Hartree) equation. Then for any $t \in [0,T]$ it holds

$$L_n \leq 2^n\,\mathrm{Tr}\Big(|x|^n\boldsymbol{\rho}^{\mathrm{in}}\Big) + C_{d,n,T}(m + \hbar)t^{\frac{n}{2}},$$

where T is given by (2.27).

Proof. We first remark that

$$\begin{aligned}[|\mathsf{p}|^2, |x|^n] &= -2i\hbar\nabla(|x|^n)\cdot\mathsf{p} + (-i\hbar)^2\Delta(|x|^n) \\ &= -ni\hbar|x|^{n-2}(2x\cdot\mathsf{p} - i\hbar(d+n-2)).\end{aligned}$$

Therefore, by defining $N_n := \mathrm{Tr}(|x|^n\boldsymbol{\rho})$, we can compute

$$\begin{aligned}\frac{\mathrm{d}N_n}{\mathrm{d}t} &= \frac{1}{i\hbar}\,\mathrm{Tr}\left(\left[|x|^n, \frac{|\mathsf{p}|^2}{2} + V\right]\boldsymbol{\rho}\right) \\ &= \frac{1}{2i\hbar}\,\mathrm{Tr}\Big([|x|^n, |\mathsf{p}|^2]\boldsymbol{\rho}\Big) \\ &= n\,\mathrm{Tr}\Big(|x|^{n-2}(2x\cdot\mathsf{p} - i\hbar(d+n-2))\boldsymbol{\rho}\Big)\,.\end{aligned}$$

Then, by Hölder's inequality for the trace, it holds

$$\frac{\mathrm{d}N_n}{\mathrm{d}t} \leq nN_n^{1-\frac{1}{n}}\left(M_n^{\frac{1}{n}} + \hbar n(d+n-2)M_0^{\frac{1}{n}}\right). \tag{2.29}$$

Hence by using the bound (2.28), we deduce that for any $t \in [0, T]$, it holds

$$\frac{\mathrm{d}N_n}{\mathrm{d}t} \leq nC_T N_n^{1-\frac{1}{n}}.$$

where $C_T = m^{\frac{1}{n}} + \hbar n(d+n-2)M_0^{\frac{1}{n}}$. By Gronwall's Lemma, we deduce

$$N_n(t) \leq \left((N_n^{\mathrm{in}})^{\frac{1}{n}} + C_T t\right)^n.$$

Finally, by Lemma 2.11 and convexity of $x \mapsto |x|^n$, we obtain

$$\begin{aligned} L_n &\leq 2N_n + C_n(t^n M_n + |\hbar t|^{\frac{n}{2}} M_0) \\ &\leq 2^n \left(N_n^{\mathrm{in}} + (C_T^n + C_n m)t^n + C_n |\hbar t|^{\frac{n}{2}} M_0\right) \\ &\leq 2^n \left(N_n^{\mathrm{in}} + t^{\frac{n}{2}} \left((C_T^n + C_n m)T^{\frac{n}{2}} + C_n \hbar M_0\right)\right), \end{aligned}$$

which yields the result. □

Proof of Theorem 2.1. Since $\mathfrak{b} < \frac{d}{2}$, we have $a := \frac{d}{\mathfrak{b}} - 1 > 1$. Thus, by Gronwall's Lemma and Proposition 2.10, for any $t > \tau > 0$ we obtain

$$\begin{aligned} L_n(t)^{-a/n} &\geq L_n(\tau)^{-a/n} + \frac{1}{A}\left(\frac{1}{t^{a-1}} - \frac{1}{\tau^{a-1}}\right) \\ &\geq L_n(\tau)^{-a/n} - \frac{1}{A\tau^{a-1}}, \end{aligned}$$

where

$$A = \left(1 - \frac{1}{a}\right) \frac{n}{C\|\nabla K\|_{L^{\mathfrak{b},\infty}} M_0^{\Theta_0} \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r}^{\frac{r'}{\mathfrak{b}}}}.$$

Combining the above inequality with Proposition 2.12, we know that there exists T such that for any $\tau \in (0, T]$ and $t > 0$, it holds

$$L_n(t) \leq \left(\left(2^n N_n^{\mathrm{in}} + C_T \tau^{\frac{n}{2}}\right)^{-\frac{a}{n}} - \frac{1}{A\tau^{a-1}}\right)^{-n/a},$$

where $C_T = C_{T, M_4^{\mathrm{in}}, M_0}$. Since $\mathfrak{b} > \frac{d}{3}$, we have $2 < a' = \frac{a}{a-1}$. Now let $\tau_0 := \min\left(T, (AC_T^{\frac{-a}{n}})^{\frac{2}{2-a}}\right)$ and $\mathcal{C} := 2^{-n}(A^{\frac{n}{a}} \tau_0^{\frac{n}{a'}} - C_T \tau_0^{\frac{n}{2}})$. We remark that $\mathcal{C} \geq 0$ since

$$\tau_0 \geq (AC_T^{\frac{-a}{n}})^{\frac{2}{2-a}} \implies C_T \tau_0^{\frac{n}{2} - \frac{n}{a'}} \leq A^{\frac{n}{a}} \implies \mathcal{C} \geq 0.$$

Taking $\tau = \tau_0$ and $N_n^{\mathrm{in}} < \mathcal{C}$, we obtain that

$$\begin{aligned} C_{T, N_n^{\mathrm{in}}, M_4^{\mathrm{in}}, M_0} &:= (2^n N_n^{\mathrm{in}} + C_T \tau_0^{\frac{n}{2}})^{-\frac{a}{n}} - \frac{1}{A\tau_0^{a-1}} \\ &> (\mathcal{C} + C_T \tau_0)^{-\frac{a}{n}} - \frac{1}{A\tau_0^{a-1}} = 0. \end{aligned}$$

We deduce that for any $t > 0$

$$L_n(t) < C_{T, N_n^{\mathrm{in}}, M_4^{\mathrm{in}}, M_0}^{-\frac{n}{a}},$$

which proves the result. □

2.3.3 Application to the semiclassical limit

Remark that we have the following corollary of Lemma 2.11, which proves in particular that for a fixed $(t,\hbar) \in \mathbb{R}_+^* \times \mathbb{R}_+$, $\mathcal{L}^1(1+|x|^n+|t\mathsf{p}|^n)$ and $\mathcal{L}^1(1+|x|^n+|x-t\mathsf{p}|^n)$ have equivalent norms.

Corollary 2.13. *Let $n \in 2\mathbb{N}$ and assume $\hbar < 1$. Then for any $\varepsilon > 0$, there exists $C_\varepsilon = C_{n,\varepsilon} > 0$ such that for any operator $\boldsymbol{\rho} \geq 0$ and any $t \geq 0$*

$$t^n \operatorname{Tr}(|\mathsf{p}|^n \boldsymbol{\rho}) \leq (1+\varepsilon) \operatorname{Tr}(|x|^n \boldsymbol{\rho}) + C_\varepsilon \operatorname{Tr}\Big(\Big(|x-t\mathsf{p}|^n + |\hbar t|^{\frac{n}{2}}\Big)\,\boldsymbol{\rho}\Big).$$

***Proof*.** We just remark that since $\tilde{\mathsf{S}}_t^{-1} = \tilde{\mathsf{S}}_{-t}$ and by the properties (2.15) of $\tilde{\mathsf{S}}$, we have for any $t \in \mathbb{R}$

$$|t|^n \operatorname{Tr}(|\mathsf{p}|^n \boldsymbol{\rho}) = |t|^n \operatorname{Tr}\Big(\tilde{\mathsf{S}}_{1/t}(|\mathsf{p}|^n)\tilde{\mathsf{S}}_{1/t}\boldsymbol{\rho}\Big) = \operatorname{Tr}\Big(|x-t\mathsf{p}|^n \tilde{\mathsf{S}}_{1/t}\boldsymbol{\rho}\Big).$$

Therefore, using Lemma 2.11, we obtain

$$\begin{aligned}
|t|^n \operatorname{Tr}(|\mathsf{p}|^n \boldsymbol{\rho}) &\leq \operatorname{Tr}\Big(((1+\varepsilon)|x|^n + C_\varepsilon\left(|t\mathsf{p}|^n + |\hbar t|^{\frac{n}{2}}\right))\tilde{\mathsf{S}}_{1/t}\boldsymbol{\rho}\Big) \\
&\leq \operatorname{Tr}\Big(\tilde{\mathsf{S}}_{-1/t}\left((1+\varepsilon)|x|^n + C_\varepsilon\left(|t\mathsf{p}|^n + |\hbar t|^{\frac{n}{2}}\right)\right)\boldsymbol{\rho}\Big) \\
&\leq \operatorname{Tr}\Big(\left((1+\varepsilon)|x|^n + C_\varepsilon\left(|x+t\mathsf{p}|^n + |\hbar t|^{\frac{n}{2}}\right)\right)\boldsymbol{\rho}\Big).
\end{aligned}$$

Replacing t by $-t$ and taking $t \geq 0$ yields the result. □

From the above corollary and the result of Theorem 2.1, we obtain the following bounds

Proposition 2.14. *Under the hypotheses of Theorem 2.1, it holds*

$$\begin{aligned}
N_n &\leq N_n^{\mathrm{in}} + C(t^{n+\varepsilon} + t) \\
M_n &\leq C(1+t^\varepsilon),
\end{aligned}$$

where the constants $C > 0$ involved depends on ε, $\|\nabla K\|_{L^{\flat,\infty}}$, M_0, M_n^{in}, $\|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r}$, d, n and r.

***Proof*.** We go back to equation (2.29) which says that for $N_n = N_n(t)$ we have

$$\frac{\mathrm{d}N_n}{\mathrm{d}t} \leq n N_n^{1-\frac{1}{n}}\left(M_n^{\frac{1}{n}} + \hbar C_{d,n} M_0^{\frac{1}{n}}\right). \tag{2.30}$$

where $C_{d,n} = n(d+n-2)$. Using Corollary 2.13 to bound M_n yields for any $\varepsilon > 0$ and any $t > \tau > 0$,

$$\begin{aligned}
\frac{\mathrm{d}N_n}{\mathrm{d}t} &\leq n N_n^{1-\frac{1}{n}}\left(\left((1+\varepsilon^n)N_n + C_\varepsilon(L_n + |\hbar t|^{\frac{n}{2}})\right)^{\frac{1}{n}} t^{-1} + \hbar C_{d,n} M_0^{\frac{1}{n}}\right) \\
&\leq n(1+\varepsilon)N_n t^{-1} + n N_n^{1-\frac{1}{n}}\left(\left(C_\varepsilon(L_n\tau^{-n} + |\hbar\tau^{-1}|^{\frac{n}{2}})\right)^{\frac{1}{n}} + \hbar C_{d,n} M_0^{\frac{1}{n}}\right) \\
&\leq n(1+2\varepsilon)N_n t^{-1} + C_{d,n,\varepsilon}\left(L_n^{\frac{1}{n}}\tau^{-1} + |\hbar\tau^{-1}|^{\frac{1}{2}} + \hbar M_0^{\frac{1}{n}}\right)^n,
\end{aligned}$$

where we used the triangle inequality for $x \mapsto |x|^{\frac{1}{n}}$ and Young's inequality $ab \leq \varepsilon a^p + C_\varepsilon b^{p'}$. Since L_n is uniformly bounded in time by Theorem 2.1, we obtain that

$$B := C_{d,n,\varepsilon} \left(L_n^{\frac{1}{n}} \tau^{-1} + |\hbar\tau^{-1}|^{\frac{1}{2}} + \hbar M_0^{\frac{1}{n}} \right)^n,$$

is also uniformly bounded in time. Therefore, for any $\varepsilon > 0$ and $t > \tau$, Gronwall's inequality yields

$$N_n(t) \leq N_n(\tau) + \frac{B\tau^{1-n-\varepsilon}}{n+\varepsilon-1} t^{n+\varepsilon}.$$

However, since as previously stated we know by [133, Proof of Theorem 3] that M_n is bounded on $[0,T]$ for a short time T depending of M_n^{in}. By inequality (2.30), it implies that $N_n(t) \leq N_n^{\mathrm{in}} + C_T t$ for any $t \in [0,T]$. Therefore, taking $\tau = T$ finally yields for any $\varepsilon \in (0,1)$ and $t \geq 0$

$$N_n(t) \leq N_n^{\mathrm{in}} + C_{B,T,n,\varepsilon}(t^{n+\varepsilon} + t).$$

The bound on M_n is then an immediate consequence of Corollary 2.13 for large times and the fact that M_n is bounded on $[0,T]$. □

In fact, it is sufficient to use the condition of smallness of moments for $n = 4$ to get a global propagation of higher moments as soon as $\mathfrak{b}_4 > \mathfrak{b}_n$ (which corresponds to $r > \frac{d}{d-1}$), which leads to the following

Proposition 2.15. *Under the condition of Theorem 2.3, $M_n \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ and more precisely there exists $c_n = c_{d,n,r} > 0$ and $C > 0$ depending on the initial conditions such that*

$$M_n \leq C(1 + t^{c_n}).$$

Proof of theorems 2.3. Since $\mathfrak{b} \geq \mathfrak{b}_4$ and $\mathfrak{b} \geq \frac{d}{3}$, we can use Proposition 2.14 for $n = 4$, and deduce

$$M_4 \leq C(1 + t^\varepsilon),$$

for a given $C > 0$. This already proves the result in the case $n = 4$, so that we assume now that $n \geq 6$. Then, we use formula (44) from [133], which reads

$$\frac{\mathrm{d}M_n}{\mathrm{d}t} \leq C_{d,r,n} C_K \|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^r}^{\Theta_2} M_{n-2}^{\Theta_0} M_n^{\Theta}, \tag{2.31}$$

with

$$\Theta = 1 + \frac{n-1}{2}\left(\frac{\mathfrak{b}_{n-2}}{\mathfrak{b}} - 1\right).$$

$$\Theta_0 = (1-\varepsilon)\left(\frac{3}{2} - \frac{r'}{p'_{n-2}}\right)$$

$$\Theta_2 = \frac{3}{2} - \Theta_1 - \Theta_0,$$

where

$$\varepsilon = \frac{nr'+d}{(n-2)r'+3d}\left(\frac{(n-2)r'+d}{\mathfrak{b}} - (n-2)\right).$$

In particular, since $r \geq d'$, $\mathfrak{b}_n$ is a non-increasing sequence and we deduce that for any $n \geq 6$, $\mathfrak{b} \geq \mathfrak{b}_4 \geq \mathfrak{b}_{n-2}$, which implies that $\Theta \leq 1$. We then obtain inequality (2.6) by Gronwall's Lemma and by recurrence over $n \in 2\mathbb{N}$. From this bound, formula (2.7) about N_n can be deduced by using again inequality (2.29) and Gronwall's Lemma. Finally, since we know by Theorem 2.1 that L_4 is bounded, the asymptotic behavior of ρ in formula (2.8) is a consequence of Corollary 2.7. □

Proof of Theorem 2.5. The hypotheses of Theorem 2.3 are fulfilled with $r = \infty$, thus we deduce that

$$M_n \leq C(1+t^c).$$

Therefore, we can use [133, Proposition 5.3], which tells us that for any $(n_0, n) \in (2\mathbb{N})^2$ verifying $d < n_0 \leq (1-\frac{1}{\mathfrak{b}})n + 1 - \frac{d}{\mathfrak{b}}$, it holds

$$\begin{aligned}
c_{d,n_0}\|\rho(t)\|_{L^\infty} &\leq \|\boldsymbol{\rho}(t)\|_{\mathcal{L}^\infty(\mathsf{p}_{\mathrm{i}}^{n_0})} \\
&\leq 2^{n_0}\left(\|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^\infty(\mathsf{p}_{\mathrm{i}}^{n_0})} + \tilde{C}_{\boldsymbol{\rho}^{\mathrm{in}}}\left(t + \int_0^t M_n^{1-\frac{1}{\mathfrak{b}}}\right)^{n_0}\right) \\
&\leq 2^{n_0}\left(\|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^\infty(\mathsf{p}_{\mathrm{i}}^{n_0})} + \tilde{C}_{\boldsymbol{\rho}^{\mathrm{in}}}\left((1+C)t + Ct^{1+\frac{c}{\mathfrak{b}'}}\right)^{n_0}\right) \\
&\leq 2^{n_0}\left(\|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^\infty(\mathsf{p}_{\mathrm{i}}^{n_0})} + \tilde{C}_{\boldsymbol{\rho}^{\mathrm{in}}}(1+C)^{n_0}2^{n_0-1}t^{n_0}\left(1+t^{\frac{cn_0}{\mathfrak{b}'}}\right)\right),
\end{aligned}$$

where $\tilde{C}_{\boldsymbol{\rho}^{\mathrm{in}}} = (4^{n_0}C_{d,n}\|\nabla K\|_{L^{\mathfrak{b}}}(1+M_0))^{n_0}\|\boldsymbol{\rho}^{\mathrm{in}}\|_{\mathcal{L}^\infty}^{1+\frac{n_0}{\mathfrak{b}}}$. This proves (2.9). As in [105, Section 4], we then define the time dependent coupling $\boldsymbol{\gamma} = \boldsymbol{\gamma}(t,z)$ with $z = (x,\xi)$ as the solution to the Cauchy problem

$$\partial_t\boldsymbol{\gamma} = (-v\cdot\nabla_x - E\cdot\nabla_\xi)\boldsymbol{\gamma} + \frac{1}{i\hbar}[H,\boldsymbol{\gamma}],$$

with initial condition $\boldsymbol{\gamma}^{\mathrm{in}} \in \mathcal{C}(f^{\mathrm{in}}, \boldsymbol{\rho}^{\mathrm{in}})$. As proved in [105, Lemma 4.2], $\boldsymbol{\gamma} \in \mathcal{C}(f(t), \boldsymbol{\rho}(t))$. We also define

$$\mathcal{E}_\hbar = \mathcal{E}_\hbar(t) := \int_{\mathbb{R}^{2d}} \mathrm{Tr}(\mathbf{c}_\hbar(z)\boldsymbol{\gamma}(z))\,\mathrm{d}z.$$

Then, by [133, Proof of Proposition 6.3], we obtain

$$\begin{aligned}
W_{2,\hbar}(f(t),\boldsymbol{\rho}(t)) &\leq \max\left(\sqrt{d\hbar}, \mathcal{E}_\hbar\right) \\
\frac{\mathrm{d}\ln(\mathcal{E}_\hbar)}{\mathrm{d}t} &\leq 2\lambda + \ln(\mathcal{E}_\hbar)/\sqrt{2},
\end{aligned}$$

with

$$\begin{aligned}
\lambda &= 1 + C_d\|\nabla K\|_{B^1_{1,\infty}}(\|\rho_f\|_{L^\infty} + \|\rho\|_{L^\infty}) \\
&\leq C^{\mathrm{in}}(1+t^{n_0(1+c/\mathfrak{b}')}),
\end{aligned}$$

where $C^{\text{in}} = 1 + C_d \|\nabla K\|_{B^1_{1,\infty}} \sup_t(\|\rho_f(t)\|_{L^\infty} + \|\rho(t)\|_{L^\infty})/(1 + t^{n_0(1+c/\mathfrak{b}')})$. This yields

$$\begin{aligned}\ln(\mathcal{E}_\hbar) &\leq \ln(\mathcal{E}_\hbar(0))e^{t/\sqrt{2}} + 2\int_0^t \lambda(s) e^{(t-s)/\sqrt{2}} \,\mathrm{d}s \\ &\leq \ln(\mathcal{E}_\hbar(0))e^{t/\sqrt{2}} + \lambda(e^{t/\sqrt{2}} - 1),\end{aligned}$$

with $\lambda = C_d C^{\text{in}}$. Therefore, as in [133, Proof of Proposition 6.3], we obtain

$$W_{2,\hbar}(f(t), \boldsymbol{\rho}(t)) \leq \max\left(\sqrt{d\hbar},\, W_{2,\hbar}(f^{\text{in}}, \boldsymbol{\rho}^{\text{in}})^{e^{t/\sqrt{2}}} e^{\lambda(e^{t/\sqrt{2}}-1)}\right),$$

which ends the proof. □

Appendix A

Some Results of Functional Analysis

A.1 Besov Spaces

We recall that a possible definition of Besov spaces (see e.g. [14]) can be done by defining the following norm

$$\|u\|_{B^s_{p,r}} = \left\| \left(2^{sj} \|\Delta_j u\|_{L^p}\right)_{j\in\mathbb{Z}} \right\|_{\ell^r}, \tag{A.1}$$

where Δ_j is defined by

$$\begin{aligned} \Delta_j u &= 0 && \text{when } j \leq -2 \\ \Delta_{-1} u &= \hat{\chi} * u \\ \Delta_j u &= \mathcal{F}_y(\varphi(2^{-j}y)) * u && \text{when } j \geq 0, \end{aligned}$$

with

$$\begin{aligned} &\chi \in C_c^\infty(B(0,4/3),[0,1]) \\ &\varphi \in C_c^\infty\left(B\left(0,8/3\right) \backslash B\left(0,3/4\right),[0,1]\right) \\ &\chi + \sum_{j\geq 0} \varphi(2^{-j}\cdot) = 1. \end{aligned} \tag{A.2}$$

We also define the space of log-Lipschitz functions by defining the norm

$$\|u\|_{LL} = \sup_{|x-y|\in(0,1)} \left(\frac{|u(x)-u(y)|}{|x-y|\left(1+|\ln(|x-y|)|\right)} \right),$$

for measurable functions u vanishing at infinity. We have the following properties of Besov spaces

Proposition A.16.

$$B^1_{\infty,\infty} \subsetneq LL. \tag{A.3}$$

If K is the Coulomb potential such that $\Delta K = \delta_0$, then we get

$$|\nabla K| = \frac{C}{|x|^{d-1}} \in B^1_{1,\infty}. \tag{A.4}$$

If $v \in L^\infty$ and $u \in B^1_{1,\infty}$, then

$$\|u * v\|_{B^1_{\infty,\infty}} \leq \|u\|_{B^1_{1,\infty}} \|v\|_{L^\infty} \tag{A.5}$$

$$\|u * v\|_{\dot{H}^1} \leq C\|u\|_{B^1_{1,\infty}} \|v\|_{L^2}. \tag{A.6}$$

Proof. The proof of (A.3) and (A.4) can be found for example in [14, Chapter 2]. To prove (A.5), we remark that since Δ_j is a convolution by a smooth and rapidly decaying function, $\Delta_j(u*v) = \Delta_j(u)*v$. By Hölder's inequality, we deduce the following inequality

$$\begin{aligned}\|u * v\|_{B^1_{\infty,\infty}} &= \left\| \left(2^j \|\Delta_j u * v\|_{L^\infty}\right)_{j\in\mathbb{Z}} \right\|_{\ell^\infty} \\ &\leq \|v\|_{L^\infty} \left\| \left(2^j \|\Delta_j u\|_{L^1}\right)_{j\in\mathbb{Z}} \right\|_{\ell^\infty} = \|u\|_{B^1_{1,\infty}} \|v\|_{L^\infty}.\end{aligned}$$

To prove (A.6), we use the Fourier definition of $\dot{H}^1$ and the fact the Fourier transform is an isometry on L^2 to obtain

$$\|u * v\|_{\dot{H}^1} \leq C\||y|\hat{u}(y)\hat{v}(y)\|_{L^2_y} \leq C\||y|\hat{u}(y)\|_{L^\infty_y} \|v\|_{L^2}.$$

Then by using the fact that $\varphi(2^{-j}y) > 0 \Leftrightarrow |y| \in 2^j[3/4, 8/3]$, we obtain the existence of $j_y \geq -2$ such that $\varphi(2^{-j}y) = 0$ for any $j \notin \{j_y - 1, j_y, j_y + 1\}$ (If $j_y = -2$, then it means that $\chi(y) > 0$). Then, by (A.2), we get

$$\begin{aligned}\|y\hat{u}(y)\|_{L^\infty} &= \left\| \left(\chi(y) + \sum_{j\geq 0} \varphi(2^{-j}y)\right) |y|\hat{u}(y) \right\|_{L^\infty_y} \\ &\leq C \left\| \sum_{k=-1}^{1} 2^{j_y+k}\, \mathcal{F}(\Delta_{j_y+k}u)(y) \right\|_{L^\infty} \\ &\leq C \sup_{j\in\mathbb{Z}} \left(2^j \|\mathcal{F}(\Delta_j u)\|_{L^\infty}\right) \leq C \left\| \left(2^j \|\Delta_j u\|_{L^1}\right)_{j\in\mathbb{Z}} \right\|_{\ell^\infty}.\end{aligned}$$

Therefore, by the definition (A.1), we obtain (A.6). □

A.2 Wasserstein distances

We recall the definition of the classical Wasserstein-(Monge-Kantorovich) distances between two probability measures $(\mu_0, \mu_1) \in \mathcal{P}(X)^2$ on a given separable Banach space X. We first define the notion of coupling by saying that $\gamma \in \mathcal{P}(X^2)$ is a coupling of μ_0 and μ_1 when

$$(\pi_1)_\# \gamma = \mu_0 \text{ and } (\pi_2)_\# \gamma = \mu_1,$$

where π_1 and π_2 are respectively the projection on the first and second variable and $\pi_\# \gamma$ denotes the push-forward of the measure γ by the map π. In other words

$$\forall \varphi \in C_0(X), \int_{X^2} \varphi(x)\gamma(\mathrm{d}x\,\mathrm{d}y) = \int_X \varphi(x)\mu_0(\mathrm{d}x).$$

We denote by $\Pi(\mu_0, \mu_1)$ the set of couplings of μ_0 and μ_1. Then we define the Wasserstein-(Monge-Kantorovich) distance in the following way

$$W_p(\mu_0, \mu_1) := \left(\inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{X^2} \|x - y\|_X^p \gamma(\mathrm{d}x\,\mathrm{d}y) \right)^{\frac{1}{p}}. \tag{A.7}$$

The existence of a minimizer is well known and we refer for example to the books [209] or [191] for more properties of these distances.

The following proposition may be classical but we prove it for the sake of completeness

Proposition A.17. *Let* $(f_0, f_1) \in \mathcal{P}(\mathbb{R}^{2d})^2$ *and for* $\mathrm{i} \in \{0, 1\}$, *let* $\rho_\mathrm{i} = (\pi_1)_\# f_\mathrm{i}$. *Then*

$$W_2(\rho_0, \rho_1) \leq W_2(f_0, f_1).$$

Proof. Let $\gamma \in \mathcal{P}(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$ be the optimal transport plan from f_0 to f_1 and define $\gamma_\rho = (\pi_{1,3})_\# \gamma$ by

$$\forall \varphi \in C_0(\mathbb{R}^{2d}), \int_{\mathbb{R}^{2d}} \varphi(x, y)\gamma_\rho(\mathrm{d}x\,\mathrm{d}y) := \int_{\mathbb{R}^{4d}} \varphi(x, y)\gamma(\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}\eta).$$

Then for any $\varphi \in C_0$, since the first marginal of γ is f_0,

$$\begin{aligned} \int_{\mathbb{R}^{2d}} \varphi(x)\gamma_\rho(\mathrm{d}x\,\mathrm{d}y) &= \int_{\mathbb{R}^{4d}} \varphi(x)\gamma(\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}\eta) \\ &= \int_{\mathbb{R}^{2d}} \varphi(x) f_0(\mathrm{d}x\,\mathrm{d}\xi) \\ &= \int_{\mathbb{R}^{d}} \varphi(x)\rho_0(\mathrm{d}x). \end{aligned}$$

Hence, the first marginal of γ_ρ is ρ_0. In the same way, the second marginal of γ_ρ is ρ_1, and we deduce that $\gamma_\rho \in \Pi(\rho_0, \rho_1)$. Next, let $(\varphi_n)_{n\in\mathbb{N}} \in (C_0(\mathbb{R}^{2d}) \cap L^1(\gamma_\rho))^{\mathbb{N}}$ be an increasing sequence of non-negative functions converging pointwise to $(x, y) \mapsto |x-y|^2$. By definition of γ_ρ, for any $n \in \mathbb{N}$, $\varphi \in L^1(\gamma)$. Therefore, by the monotone convergence theorem,

$$\begin{aligned} \int_{\mathbb{R}^{2d}} |x - y|^2 \gamma_\rho(\mathrm{d}x\,\mathrm{d}y) &= \lim_{n\to\infty} \int_{\mathbb{R}^{2d}} \varphi_n(x, y)\gamma_\rho(\mathrm{d}x\,\mathrm{d}y) \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^{4d}} \varphi_n(x, y)\gamma(\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}\eta) \\ &= \int_{\mathbb{R}^{4d}} |x - y|^2\, \gamma(\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}\eta) \\ &\leq \int_{\mathbb{R}^{4d}} (|x - y|^2 + |\xi - \eta|^2)\gamma(\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}\eta) = W_2(f_0, f_1)^2. \end{aligned}$$

By definition (A.7), we deduce

$$W_2(\rho_0, \rho_1)^2 \leq \int_{\mathbb{R}^{2d}} |x - y|^2 \gamma_\rho(\mathrm{d}x\,\mathrm{d}y) \leq W_2(f_0, f_1)^2,$$

which proves the result. □

Part II

Asymptotic Behaviour of Kinetic Models

Chapter 3

Fractional Fokker-Planck Equation with General Confinement Force

Abstract

This chapter studies a Fokker-Planck type equation of fractional diffusion with conservative drift

$$\partial_t f = \Delta^{\frac{\alpha}{2}} f + \mathrm{div}(Ef),$$

where $\Delta^{\frac{\alpha}{2}}$ denotes the fractional Laplacian and E is a confining force field. The main interest of the present chapter is that it applies to a wide variety of force fields with a polynomial growth at infinity.
We first prove the existence and uniqueness of a solution in weighted Lebesgue spaces depending on E under the form of a strongly continuous semigroup. We also prove the existence and uniqueness of a stationary state, by using an appropriate splitting of the fractional Laplacian and by proving a weak and strong maximum principle.
We then study the rate of convergence to equilibrium of the solution. The semigroup has a property of regularization in fractional Sobolev spaces, as well as a gain of integrability and positivity which we use to obtain polynomial or exponential convergence to equilibrium in weighted Lebesgue spaces.

Résumé

Ce chapitre étudie une équation de type Fokker-Planck de diffusion fractionnaire avec un drift conservatif

$$\partial_t f = \Delta^{\frac{\alpha}{2}} f + \mathrm{div}(Ef),$$

où $\Delta^{\frac{\alpha}{2}}$ est le Laplacien fractionnaire et E est un champs de force confinant. L'intérêt principal de ce chapitre est le fait qu'il s'applique à une large classe de champs de force avec une croissance polynomiale à l'infini et sans connaissance précise de l'état stationnaire.
On montre d'abord l'existence et l'unicité d'une solution dans des espaces de Lebesgue à poids dépendant de E, solution qui s'écrit sous la forme d'un semi-groupe continu. On montre aussi l'existence et l'unicité d'un état stationnaire en utilisant un découpage approprié du Laplacien fractionnaire et en montrant un principe du maximum fort.
On étudie enfin le taux de convergence vers l'état stationnaire. Le semi-groupe régularise dans des espaces de Sobolev fractionnaire et apporte aussi un gain d'intégrabilité et de positivité, que l'on utilise pour obtenir des taux de convergence polynomiaux ou exponentiels vers l'état d'équilibre dans des espaces de Lebesgue à poids.

Contents

3.1 Introduction

3.1.1 Presentation of the equation and preceding work

We consider the homogeneous fractional Fokker-Planck Equation

$$\partial_t f = \mathsf{L} f := I(f) + \operatorname{div}(Ef), \tag{FFP}$$

where $f = f(t,x)$ with $x \in \mathbb{R}^d$, E is a given force field with polynomial growth at infinity and

$$I = \Delta^{\frac{\alpha}{2}} \text{ with } \alpha \in (0,2)$$

is the fractional Laplacian.

The fractional Laplacian is a generalization of the Laplacian that can be seen as the opposite of a fractional iteration of the positive operator $-\Delta$. It can be defined for any nice function f through its Fourier transform by

$$\widehat{I(f)} = -|2\pi\xi|^{\alpha}\widehat{f}. \tag{3.1}$$

Alternatively, it is also defined up to a constant depending on α and d for sufficiently smooth functions f by the following integral expression (see e.g. [138, Chapter 1, §1])

$$I(f) = \mathrm{vp}\int_{\mathbb{R}^d} \frac{f(y)-f(x)}{|y-x|^{d+\alpha}}\,\mathrm{d}y, \tag{3.2}$$

where vp indicates that it is a principal value when $\alpha \geq 1$.

It can also be defined as the infinitesimal generator of a Levy process. A probabilistic point of view about fractional diffusion can for example be found in [128]. The integral representation can be seen in the perspective of the dynamic associated with this Levy process as it represents the fact that particles will jump from x to y proportionally to the difference of value of f, from the high to the low densities, and proportionally to the inverse of a power of the distance. It highlights the non-local behavior of this operator.

The fractional Laplacian is in our case in competition with the force field E. For $\alpha < 1$, the effect of the force field will be stronger in small scales, resulting in possibly discontinuous solutions (see for example [198]). We restrict ourselves to a force field with at most polynomial growth at infinity.

Physically, this equation can be first seen as the space homogeneous version of the kinetic fractional Fokker-Planck equation

$$\partial_t f + v\cdot\nabla_x f = \mathsf{L} f,$$

where $f = f(t,x,v)$ in this case. It is a particular case of the Linear Boltzmann equation which models the behavior of particles interacting with a background medium and for which the general scattering operator can be written

$$\mathsf{L}_\sigma f = \int_{\mathbb{R}^d} \sigma(x,v,v_*)\,(f(v_*)-f(v))\,\mathrm{d}v_*.$$

However, it has been shown in [161] that the diffusive limit of such equations in the case of heavy-tailed distributions of speed is a fractional heat equation in the x variable. See also [160, 66, 65, 141]. Such heavy-tailed distributions can be found in astrophysical plasmas, economy, granular gases and mixtures of gases (see [161] and references therein). As proved in [2], the (FFP) equation can then be derived by adding a force field to the above kinetic equation and is therefore the limit of the following fractional Vlasov-Fokker-Planck equation

$$\varepsilon^{\alpha}\partial_t f + \varepsilon v\cdot\nabla_x f + \varepsilon^{\alpha-1}E(x)\cdot\nabla_v f = \Delta_v^{\frac{\alpha}{2}} f + \mathrm{div}_v(vf).$$

We mention that another reason for the recent interest about the factional Laplacian is the fact that it can also be considered as a simplified version of the Boltzmann linearized operator, see for example [75, 172, 165, 166, 208, 53, 52, 124]. It was for example used extensively in [126] and in [196] to retrieve Harnack's inequalities and regularity for the Boltzmann equation without cutoff.

3.1.2 Main results

In all this chapter, we will denote by $d \in \mathbb{N}^*$ the dimension of the space for the space variable, $\Omega \subset \mathbb{R}^d$ will be an open subset, μ a measure (or its identification to a Lebesgue measurable function when it is absolutely continuous with respect to the Lebesgue measure) and m a nonnegative weight function which will often be of the form $\langle x\rangle^k$ for $k \in \mathbb{R}$ where $\langle x\rangle = \sqrt{1+|x|^2}$. We will often denote by C constants whose exact value have no importance, or write for example C_a when we want to emphasize that the constant depends on a, but also use the following notations

$$\begin{aligned} a \lesssim b &\overset{\text{def}}{\Leftrightarrow} \exists C > 0,\ a \leq Cb \\ a \simeq b &\overset{\text{def}}{\Leftrightarrow} a \lesssim b \text{ and } b \lesssim a. \end{aligned}$$

Notice that f and g will usually denote functions of time and space while u and w will usually only depend on the space variable x. Moreover, $q = p' := \frac{p}{p-1}$ will denote the Hölder conjugate of p and $a \wedge b := \min(a, b)$.

We will mainly work in weighted Lebesgue spaces denoted by $L^p(m)$ for $p \in [1, \infty]$, associated to the norm

$$\|u\|_{L^p(m)} := \|um\|_{L^p}.$$

We also recall the extension of Sobolev Spaces (see [55]) to fractional order of derivatives, which can be defined through the following semi-norms, generalization of the Hölder property to the Lebesgue spaces for $s \in (0, 1)$

$$|u|^p_{W^{s,p}} := c_{s,d} \iint_{\mathbb{R}^{2d}} \frac{|u(y) - u(x)|^p}{|y - x|^{d+ps}} \,\mathrm{d}y \,\mathrm{d}x. \tag{3.3}$$

Those are Banach Spaces for the norm $\|u\|^p_{W^{s,p}} := |u|^p_{W^{s,p}} + \|u\|^p_{L^p}$. When $s = 1$, the norm can be replaced for example by $\|u\|^p_{W^{1,p}} := \|\nabla u\|^p_{L^p} + \|u\|^p_{L^p}$. See for example [205, 206, 159, 76] for a more complete study of these spaces.

We are interested here in a confining force field with polynomial growth taking typically the form

$$E = \langle x\rangle^\beta x = \nabla\left(\frac{\langle x\rangle^{\beta+2}}{\beta + 2}\right), \tag{3.4}$$

with $\beta \in \mathbb{R}$. The case $E = x = \nabla V(x)$ with $V(x) = \frac{|x|^2}{2}$ is the most studied in the literature (see for example [30, 98, 99, 207]). In this case the steady state can be computed explicitly and the equation is equivalent up to a scaling to the fractional heat equation (see for example [33]). Since our method do not use the explicit formula for E, we will always assume the following more general hypotheses for a given $\beta \in \mathbb{R}$.

Hypotheses on E:

$$|\nabla E| \lesssim \langle x\rangle^\beta \tag{3.5}$$

$$E \cdot x \gtrsim |x|^{\beta+2}. \tag{3.6}$$

Remark also that the kernel in the definition (3.2) of the fractional Laplacian, $\kappa_\alpha : z \mapsto \frac{c_{\alpha,d}}{|z|^{d+\alpha}}$, could be replaced by any symmetric kernel κ_α verifying

$$\kappa_\alpha(z) \simeq \frac{1}{|z|^{d+\alpha}}.$$

Our first result is about existence and uniqueness of a solution.

Theorem 3.1. *Let $m := \langle x \rangle^k$ with $k \in (0, \alpha \wedge 1)$. Then there exists $p_\beta > 1$ such that for all $p \in [1, p_\beta)$, if $f^{\rm in} \in L^p(m)$, there exists a unique solution*

$$f \in C^0(\mathbb{R}_+, L^p(m))$$

to the (FFP) equation such that $f(0, \cdot) = f^{\rm in}$. Moreover, L is the generator of a C^0-semigroup in $L^p(m)$.

Remark 3.2. *As it can be seen in the proof, to prove the existence of a solution, hypotheses (3.5) and (3.6) can be weakened to the existence of $k \in (0, \alpha \wedge 1)$, $r > 2$, $p > 1$ and $E \in W^{1,r}_{\rm loc} \cap L^\infty_{\rm loc}$ such that*

$$E \cdot x \;\geq\; 0 \tag{3.7}$$

$$\varphi_{m,p} \;:=\; \frac{\operatorname{div}(E)}{q} - E \cdot \frac{\nabla m}{m} \;\leq\; C. \tag{3.8}$$

In particular, it implies that we do not need to control $\|\nabla E\|_{L^\infty}$ but only $\|\operatorname{div}(E)\|_{L^\infty}$. Moreover, when $\beta \leq 0$, (3.6) or (3.7) are unnecessary.

Remark 3.3. *When (3.5) and (3.6) hold, then (3.8) holds for $\beta \leq 0$ or p smaller than a given $p_\beta \in (1, +\infty)$ which is such that*

$$\forall p \in (1, p_\beta),\ \varphi_{m,p} \;\leq\; b\, \mathbb{1}_\Omega - a \langle x \rangle^\beta, \tag{3.9}$$

for a given $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}$ and a given bounded set Ω. This relation is similar to the Foster-Lyapunov condition for Harris recurrence (see [162, 16, 113, 85]). When E verifies hypotheses (3.5) and (3.6), then the value of p_β is given by

$$p_\beta = \frac{C_1}{C_1 - C_0 k},$$

where $C_1 = \|\nabla E\|_{L^\infty(\langle x \rangle^{-\beta})}$ and $C_0 = \inf_{x \in \mathbb{R}^d} \left(\frac{E \cdot x}{|x|^{\beta+2}} \right)$. In particular, when E takes the form (3.4), we obtain $p_\beta = 1 + \frac{k}{d+\beta-k}$.

This result generalizes the results obtained by Wei and Tian in [211], where the existence was proved for divergence-bounded force fields. The a priori estimates on weighted spaces, from where come the relations between E and p, have been already used in the case of the classical Fokker-Planck equation (for example by Gualdani and al in [111]).

Theorem 3.4. *Let $m := \langle x \rangle^k$ with $k \in (0, \alpha \wedge 1)$ and $f \in L^1(m)$ be a solution to the (FFP) equation. Then there exists $p_\beta > 1$ such that for any $t > 0$, $f(t)$ is immediately in all $L^p(m)$ for $p < p_\beta$. Moreover, if $\beta \leq 0$, $f(t) \in L^\infty(m)$.*

There has been some recent interest in the regularity theory for integro-differential equations. In [50, 194, 195, 193], it is proved that under some regularity conditions on E and if $f \in L^\infty$ is the solution to (FFP), then f is actually Hölder continuous or even more regular. However, it is also proved in [197] that there can be some loss of regularity when E is not regular enough. As proved in [67] for divergence free drifts or in Proposition 3.9, we can still obtain fractional Besov or Sobolev regularity in these cases. Theorem 3.4 gives in particular the regularization from L^1 to L^∞ in the case when $E \in C^1_b$, which then allows to use the theorems cited above.

Theorem 3.5. *Assume $\beta > -\alpha$ and $m = \langle x \rangle^k$ with $0 \leq k < \alpha \wedge 1$. Then there exists $p^* > 1$ such that for any $p \in (1, p^*)$, there exists a unique $F \in L^p(m) \cap L^1_+$ of mass 1 such that*

$$\mathsf{L} F \ = \ 0.$$

This result generalizes the results obtained by Mischler and Mouhot in [167] and Kavian and Mischler in [130] where it is proved for the classical Laplacian and respectively $\beta \geq -1$ and $\beta \in (-2, -1)$. It is also close to the result obtained by Mischler and Tristani in [169] where the fractional Laplacian is replaced by integral operators with integrable kernel.

The last and main result is the following rate of convergence towards equilibrium.

Theorem 3.6. *Assume $\beta > -\alpha$ and let $m := \langle x \rangle^k$ with $0 \leq k < (\alpha \wedge 1)$. Then, if $\beta \geq 0$, there exists $a > 0$ such that for any $p \in [1, p_\beta)$,*

$$\|f - F\|_{L^p(m)} \ \lesssim \ e^{-at} \|f^{\text{in}} - F\|_{L^p(m)}.$$

If $\beta \in (-\alpha, 0)$, there exists $p^* > 1$ such that for any $p \in [1, p^*)$ and any $\bar{k} < k$, the following rate holds

$$\|f - F\|_{L^p(\bar{m})} \ \lesssim \ \langle t \rangle^{-a} \|f^{\text{in}} - F\|_{L^p(m)},$$

where $\bar{m} = \langle x \rangle^{\bar{k}}$ and $a = \frac{\bar{k} - k}{|\beta|}$ if $p > 1$. When $p = 1$, a can be any number verifying $a < \frac{\bar{k} - k}{|\beta|}$.

This result generalizes the one obtained by Wang in [210] where, following the techniques of [98], exponential convergence of the relative entropy is obtained for force fields $E \in C^1_b$ such that $\forall x \in \mathbb{R}^d, x \cdot \nabla E \cdot x \simeq |x|^2$ and the one obtained by Tristani in [207] where exponential convergence towards equilibrium is proved in $L^p(m)$ in the case $E(x) = x$. It is also the natural extension to the fractional case of the results obtained by Kavian and Mischler in [130] and Mouhot and Mischler in [167], which correspond respectively to the case $\beta \in (-2, -1)$ and $\beta \geq -1$ for the classical Laplacian. The reason of the lower bound on $\beta > -\alpha$ is due to the strong nonlocal behavior of the fractional Laplacian which seems to compensate the confining effect of the force field.

The chapter is organized as follows. The second section proves some properties of the fractional Laplacian and of the operator L which will be useful for the various results of the chapter.

Section 3.3 proves the existence and uniqueness in the weighted $L^p(m)$ spaces for $p \in (1, 2)$. We first establish the existence of a solution for an approximated problem and then use a priori estimates and compactness properties to obtain a solution to the original problem.

Following the ideas of Nash in [173], The section 3.4 of this chapter generalizes the regularization property of the semigroup associated to the (FFP) equation as established in [207]. Moreover, a gain of integrability as well as a gain of positivity are also proved, which are useful to deal with convergence without any L^∞ bound.

In section 3.5, the existence of a stationary state is proved by using an adequate splitting of the operator. It follows the general idea of writing operators as a regularizing part and a dissipative part, as explained in [111]. We then prove a weak and strong maximum principle and deduce the uniqueness of the equilibrium from the Krein-Rutman Theorem.

The sixth section deals with polynomial convergence when E is not confining enough to create a spectral gap. It uses techniques inspired from [16] by using both Foster-Lyapunov estimates introduced by Harris and developed by Meyn and Tweedie in [162] and a local Poincaré inequality. It proves the first part of Theorem 3.6.

Last section is devoted to the proof of the exponential convergence when E is strongly confining (i.e. $\beta > 0$) and follows a different approach as it replaces the use of the Poincaré inequality by the gain of positivity property, following the work of Hairer and Mattingly in [112]. It proves the second part of Theorem 3.6. Notice that similar techniques are also used to get subgeometric rates of convergence in [84].

3.2 Main inequalities

3.2.1 Preliminary results about fractional Laplacian

We first recall the standard notations that we will use on this chapter. We will denote by $\mathscr{B}(E,F)$ the space of continuous linear mappings from E to F, by $u_+ := \max(u,0)$ the positive part of u. Moreover, we will identify bounded measures on measurable sets of $\mathbb{R}^d$ with bounded radon measures $\mu \in \mathcal{M}(\Omega) := C_0(\Omega)'$ and write

$$\int u\,\mu \ := \ \int u(x)\mu(\mathrm{d}x), \qquad\qquad \mu(A) \ := \ \int_A \mu,$$

for any μ-measurable function u and μ-measurable set A. We will write the mass of a measure $\langle u\rangle_{\mathbb{R}^d} := \int_{\mathbb{R}^d} u$. We also recall that $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$ is the space of distributions on Ω. Moreover, we will not write Ω when $\Omega = \mathbb{R}^d$.

Notice that in order to simplify the presentation, we will use the following definition for the power of a scalar, $u^a := |u|^{a-1}u$ for any $a \in \mathbb{R}$, and we will use a short notation to simplify the writing of the integrals,

$$\kappa_{\alpha,*} := \kappa_\alpha(x_* - x) \qquad\qquad u := u(x) \qquad\qquad u_* := u(x_*).$$

With these notations and since $\alpha \in (0,2)$, for sufficiently smooth and decaying functions u, we can write the fractional Laplacian as a principal value

$$I(u) \ = \ \mathrm{vp}\left(\int_{\mathbb{R}^d} \kappa_{\alpha,*}\,(u_* - u)\,\mathrm{d}x_*\right) \ = \ \lim_{\varepsilon\to 0} c_{\alpha,d}\int_{|x-y|>\varepsilon} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}}\,\mathrm{d}y.$$

Remark that the principal value can be removed when $\alpha \in (0,1)$. Another useful expression

$$I(u) \ = \ c_{\alpha,d}\int_{\mathbb{R}^d} \frac{u(y)-u(x)-(y-x)\cdot\nabla u(x)\,\mathbb{1}_{B_R}(x-y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y, \tag{3.10}$$

which is valid for any $R > 0$. By duality, it can also be defined on more general spaces of tempered distributions with a growth smaller than $|x|^\alpha$ at infinity by the formula $\langle I(u),\varphi\rangle_{\mathcal{D}',\mathcal{D}} := \langle u, I(\varphi)\rangle_{I(\mathcal{D})',I(\mathcal{D})}$. In particular, we will mostly use the fractional Laplacian of weight functions of the form $m(x) = \langle x\rangle^k$ with $k < \alpha$.

Following the model of the Laplacian, we define for $p > 1$

$$\Gamma(u,w) \ := \ \int_{\mathbb{R}^d} \frac{\kappa_{\alpha,*}}{2}\,(u_* - u)\,(w_* - w)\,\mathrm{d}x_* \tag{3.11}$$

$$\mathfrak{D}_p(u) \ := \ \Gamma(u,u^{p-1}) \ \geq \ 0. \tag{3.12}$$

The first quantity can be seen as a generalization of $\nabla u\cdot\nabla w$. It is known as the "Carré du Champs" operator in Probabilities. The second can be seen as a generalization of $\left|\nabla|u|^{p/2}\right|^2$.

The quantity (3.11) comes naturally when considering the fractional Laplacian of a product of (sufficiently smooth) functions, since the following formula holds

$$I(uw) \ = \ u\,I(w) + w\,I(u) + 2\,\Gamma(u,w). \tag{3.13}$$

Moreover, we have the following integration by parts formula

$$\int_{\mathbb{R}^d} u\, I(w) \;=\; \int_{\mathbb{R}^d} I(u) w \;=\; -\int_{\mathbb{R}^d} \Gamma(u,w). \tag{3.14}$$

So that in particular, by definition (3.12)

$$\int_{\mathbb{R}^d} I\,(u)\, u^{p-1} \;=\; -\int_{\mathbb{R}^d} \mathfrak{D}_p(u) \;\leq\; 0. \tag{3.15}$$

Remark that these relations also holds when replacing $\kappa_\alpha(x-x_*)$ by a general symmetric kernel $\kappa(x,x_*)$.

It will be useful to remark that the following quantities are equivalent.

Proposition 3.1. *Let u be such that $\mathfrak{D}_p(u)$ is bounded for a given $p \in (1,\infty)$. Then*

$$\mathfrak{D}_p(u) \;\simeq\; \frac{1}{p}\, I(|u|^p) - u^{p-1}\, I(u) \tag{3.16}$$

$$\simeq\; \frac{1}{q}\, I(|u|^p) - u\, I(u^{p-1}) \tag{3.17}$$

$$\simeq\; \int_{\mathbb{R}^d} \kappa_{\alpha,*}\, |u_*^{p/2} - u^{p/2}|^2 \,\mathrm{d}x_*, \tag{3.18}$$

where we recall that $q = p'$ and $a \simeq b$ means here that a/b is bounded by above and below by positive constants depending only on p.

Remark 3.7. *These inequalities are related to some known inequalities such as the Córdoba-Córdoba inequality [72] which tells that the expression in (3.16) and (3.17) are positive, and the Stroock-Varopoulos inequality, which corresponds to the integral version of inequality (3.18) and can be written*

$$\int_{\mathbb{R}^d} \mathfrak{D}_p(u) \geq \frac{4}{pp'} \left| u^{\frac{p}{2}} \right|^2_{H^{\frac{\alpha}{2}}}.$$

Proof of Proposition 3.1. For the first line, we remark that

$$\begin{aligned}
\mathfrak{D}_p(u) &= \iint_{\mathbb{R}^{2d}} \kappa_{\alpha,*}\,(u_* - u)\left(u_*^{p-1} - u^{p-1}\right) \mathrm{d}x_*\mathrm{d}x \\
&= \iint_{\mathbb{R}^{2d}} \kappa_{\alpha,*}\; d_1(u_*/u)|u|^p \,\mathrm{d}x_*\mathrm{d}x \\
\frac{1}{p}\, I(|u|^p) - u^{p-1}\, I(u) &= \frac{1}{p} \iint_{\mathbb{R}^{2d}} \kappa_{\alpha,*}\left(|u_*|^p - |u|^p - pu^{p-1}(u_* - u)\right) \mathrm{d}x_*\mathrm{d}x \\
&= \frac{1}{p} \iint_{\mathbb{R}^{2d}} \kappa_{\alpha,*}\; d_2(u_*/u)|u|^p \,\mathrm{d}x_*\mathrm{d}x,
\end{aligned}$$

where we recall that $u^p = |u|^{p-1}u$ and we defined for any $z \in \mathbb{R}$,

$$\begin{aligned}
d_1(z) &= (z-1)(z^{p-1}-1) \geq 0 \\
d_2(z) &= |z|^p - 1 - p(z-1) \geq 0.
\end{aligned}$$

Then we remark that d_1/d_2 is a bounded positive function since it is continuous on $\mathbb{R}\backslash\{1\}$, converges to 1 when $|z| \to \infty$ and to $2/p$ when $z \to 1$. Therefore, $d_1 \simeq d_2$ and it implies (3.16). Noting that we have the following asymptotic behaviors as $z \to 1$

$$|z^{p/2} - 1|^2 \sim \frac{p^2}{4}|z-1|^2$$
$$\frac{|z|^p - 1}{q} - (z^{p-1} - 1) \sim \frac{(p-1)}{2}|z-1|^2,$$

we can then prove the other inequalities in the same way. □

Another useful result is the estimation of the growth of the fractional Laplacian of weight functions.

Proposition 3.2 (Fractional Derivation of weight functions)**.** *Let* $k \in (0, \alpha \wedge 1)$ *and* $m : x \mapsto \langle x \rangle^k$ *defined for* $x \in \mathbb{R}^d$*. Then, the following inequality holds*

$$|I(m)| \;\leq\; \frac{C}{\langle x \rangle^{\alpha - k}}, \tag{3.19}$$

where C *is of the form* $\frac{C_k\,\omega_d}{(\alpha-k)(2-\alpha)}$*. Moreover, when* $\alpha < 1$

$$\mathrm{D}^\alpha m \;\leq\; \frac{C_{\alpha,k}}{\langle x \rangle^{\alpha-k}}, \tag{3.20}$$

$$\mathrm{D}^\alpha \left(m^{-1}\right) \;\leq\; \frac{C_{\alpha,|k|}}{\langle x \rangle^{\alpha}}, \tag{3.21}$$

where $C_{\alpha,k}$ *is of the form* $\frac{C_k\,\omega_d}{(\alpha-k)(1-\alpha)}$ *and* D^α *is defined by*

$$\mathrm{D}^\alpha u \;:=\; \int_{\mathbb{R}^d} \kappa_{\alpha,*}\,|u_* - u|\,\mathrm{d}x_*. \tag{3.22}$$

Proof of Proposition 3.2. We first look at the case $\alpha \in (0,1)$ and then at the case $\alpha \in (0,2)$ which works only for $I(m)$.

Step 1. Case $\alpha \in (0,1)$**.** Let $x \in \mathbb{R}^d$ and $R > 1$. We split D^α into two parts

$$\mathrm{D}^\alpha m \;\leq\; \int_{|x-y|>R} \frac{|m(x) - m(y)|}{|x-y|^{d+\alpha}}\,\mathrm{d}y + \int_{|x-y|\leq R} \frac{|m(x) - m(y)|}{|x-y|^{d+\alpha}}\,\mathrm{d}y \;\;=: \;\mathcal{I}_1 + \mathcal{I}_2.$$

For the first part, we remark that since $k \in (0,1)$ and $\forall y \in \mathbb{R}, |\nabla \langle y \rangle| \leq 1$, we obtain

$$|\langle x \rangle^k - \langle y \rangle^k| \;\leq\; |\langle x \rangle - \langle y \rangle|^k \;\leq\; |x-y|^k.$$

It leads to

$$\mathcal{I}_1 \;\leq\; \int_{|z|>R} \frac{\mathrm{d}z}{|z|^{d+\alpha-k}} \;\leq\; \frac{\omega_d}{(\alpha-k)R^{\alpha-k}}.$$

• If $|x| \geq 1$, we take $R := |x|/2$. Then $|x|^{-1} \leq \sqrt{2}\,\langle x \rangle^{-1}$, from what we deduce

$$\mathcal{I}_1 \;\leq\; \frac{C\,\omega_d}{(\alpha-k)}\frac{1}{\langle x \rangle^{\alpha-k}}.$$

Let $y \in \mathbb{R}^d$ be such that $|x-y| < |x|/2$. For $z \in [x,y] \subset \mathbb{R}^d$, we have $|z| \geq |x| - |x-z| \geq |x|/2$. Thus, we obtain

$$\begin{aligned} |m(x)-m(y)| \;&\leq\; |x-y| \sup_{[x,y]} |\nabla m| \\ &\leq\; |x-y| \sup_{z\in[x,y]} |k\,\langle z \rangle^{k-2} z| \\ &\leq\; 2^{1-k} k\,\langle x \rangle^{k-1} |x-y|, \end{aligned} \tag{3.23}$$

where we used $|x| \leq \langle x \rangle$ and $\langle x/2 \rangle \geq \langle x \rangle/2$. It implies the following upper bound

$$\mathcal{I}_2 \;\leq\; C\,\langle x \rangle^{k-1} \int_{|z|\leq|x|/2} \frac{\mathrm{d}z}{|z|^{d+\alpha-1}} \;\leq\; \frac{C\,\omega_d}{1-\alpha}\,\langle x \rangle^{k-\alpha}.$$

• If $|x| \leq 1$, we take $R := 1$ and we deduce

$$\mathcal{I}_1 \;\leq\; \frac{\omega_d}{(\alpha-k)}.$$

Moreover, as $k\,\langle x \rangle^{k-1} \leq 1$, (3.23) gives us

$$|m(x)-m(y)| \;\leq\; |x-y|.$$

Therefore

$$\mathcal{I}_2 \;\leq\; \int_{|z|\leq 1} \frac{\mathrm{d}z}{|z|^{d+\alpha-1}} \;\leq\; \frac{\omega_d}{1-\alpha}.$$

• We end the proof of (3.20) by gathering the two parts together. Since $m \geq 1$, we get (3.21) by remarking that

$$\mathrm{D}^\alpha(m^{-1}) \;=\; \int_{\mathbb{R}^d} \kappa_{\alpha,*} \left|\frac{m_*-m}{m_* m}\right| \mathrm{d}x_* \;\leq\; \frac{\mathrm{D}^\alpha m}{m}.$$

***Step 2. Proof of* (3.19).** We use the integral representation (3.10) to change $\mathcal{I}_2$ by

$$\mathcal{I}_2 = \int_{|x-y|\leq R} \frac{|m(x)-m(y)-(x-y)\cdot\nabla m(x)|}{|x-y|^{d+\alpha}}\,\mathrm{d}y.$$

Then (3.23) is replaced by a second order Taylor inequality, which gives

$$|m(x)-m(y)-(x-y)\cdot\nabla m(x)| \;\leq\; C_k\,\langle x \rangle^{k-2} |x-y|^2.$$

The other parts of the proof are similar to the step 1. □

3.2.2 Inequalities for the generator of the semigroup

To get existence, uniqueness and additional gains of weight and regularity on the solutions to the (FFP) equation, the main inequalities are given in the following

Proposition 3.3. *Let $m = \langle x\rangle^k$ with $k \in (0,1)$ and $u \in L^p(m\,\langle x\rangle^{\beta_+})$. If $k < \alpha < 1$, the following holds*

$$\int_{\mathbb{R}^d} \mathsf{L}(u)u^{p-1}m^p + \int_{\mathbb{R}^d} \mathfrak{D}_p(um) \;\le\; \int_{\mathbb{R}^d} |u|^p m^p \left(\frac{C_k}{\langle x\rangle^{\alpha-k}} + \varphi_{m,p}\right), \tag{3.24}$$

where $\varphi_{m,p}$ is defined by (3.8) and $\mathfrak{D}_p \ge 0$ is defined by (3.12). If $kp < (\alpha \wedge 1)$, we also have

$$\int_{\mathbb{R}^d} \mathsf{L}(u)u^{p-1}m^p + C_p\int_{\mathbb{R}^d} \mathfrak{D}_p(um) \;\le\; \int_{\mathbb{R}^d} |u|^p m^p \left(\frac{C_{k,p}}{\langle x\rangle^{\alpha}} + \varphi_{m,p}\right). \tag{3.25}$$

Remark 3.8. *These inequalities are the generalization of the estimates obtained in [130, 167]. As already pointed out in introduction, $\varphi_{m,p}$ is always bounded above when $\beta \le 0$. When $\beta > 0$, there exists $p_\beta > 1$ such that $\varphi_{m,p}$ is bounded for any $p \in (1, p_\beta)$. Moreover, in this case, there exists $(a,b) \in \mathbb{R}_+^* \times \mathbb{R}$ such that*

$$\varphi_{m,p} \le b - a\,\langle x\rangle^{\beta}.$$

Inequality (3.25) is more restrictive on k since it needs $k < \alpha/p$, but it has the advantage to work for all $\alpha \in (0,2)$ and to give a second term with a smaller weight.

Lemma 3.4. *Let $m = \langle x\rangle^k$ with $|k| < \alpha \le 1$ and $u \in L^p(m\,\langle x\rangle^{(k-\alpha)/p})$. Then the following inequality holds true*

$$\left|\int_{\mathbb{R}^d} (I(mu) - m\,I(u))\,(um)^{p-1}\right| \;\le\; C_k\,\|u\|^p_{L^p(m\langle x\rangle^{(k-\alpha)/p})}. \tag{3.26}$$

Proof of Lemma 3.4. Let $j_m(u) := I(mu) - m\,I(u)$. Using the integral definition (3.2) of I, we have

$$\begin{aligned}\int_{\mathbb{R}^d} j_m(u)\,w &= \iint_{\mathbb{R}^{2d}} \kappa_{\alpha,*}\,((u_* m_* - um) - m\,(u_* - u))\,w\,\mathrm{d}x_*\mathrm{d}x\\ &= \iint_{\mathbb{R}^{2d}} \kappa_{\alpha,*}\,\frac{m_* - m}{m_*}\,(u_* m_*)\,w\,\mathrm{d}x_*\mathrm{d}x.\end{aligned}$$

Thus, by Hölder's inequality, we get

$$\left|\int_{\mathbb{R}^d} j_m(u)\,w\right| \;\le\; \left(\iint_{\mathbb{R}^{2d}} \kappa_{\alpha,*}\frac{|m_* - m|}{m_*}|u_* m_*|^p\,\mathrm{d}x_*\mathrm{d}x\right)^{\frac{1}{p}} \left(\iint_{\mathbb{R}^{2d}} \kappa_{\alpha,*}\frac{|m_* - m|}{m_*}|w|^q\,\mathrm{d}x_*\mathrm{d}x\right)^{\frac{1}{q}}.$$

By the fact that $\frac{|m_*-m|}{m_*} = \frac{|m_*^{-1}-m^{-1}|}{m^{-1}}$ and exchanging x and x_* in the first integral, we obtain

$$\left|\int_{\mathbb{R}^d} j_m(u)\,w\right| \;\le\; \left(\int_{\mathbb{R}^d} \frac{\mathrm{D}^\alpha m}{m}|um|^p\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} \frac{\mathrm{D}^\alpha (m^{-1})}{m^{-1}}|w|^q\right)^{\frac{1}{q}}.$$

where D^α is defined by (3.22). In particular, if $m = \langle x\rangle^k$ with $|k| < \alpha$, we obtain from Proposition 3.2

$$\left|\int_{\mathbb{R}^d} j_m(u)\, w\right| \;\le\; C_k \,\|u\|_{L^p(m\langle x\rangle^{-\alpha/p})}\, \|w\|_{L^q(\langle x\rangle^{(k-\alpha)/q})},$$

which implies (3.26) by taking $w = (um)^{p-1}$. □

Proof of Proposition 3.3. Let $\Phi = \frac{|\cdot|^p}{p}$ and $u \in C_c^\infty$. Then, by definition

$$\int_{\mathbb{R}^d} \mathsf{L}(u)\Phi'(u)m^p \;=\; \int_{\mathbb{R}^d} I(u)\,\Phi'(u)m^p + \mathrm{div}(Eu)\,\Phi'(u)m^p.$$

Let first focus on the term containing the force field E. We expand the divergence of the product, use the fact that $\Phi'(u)\nabla u = \nabla\Phi(u)$ and integrate by parts the second term to find

$$\int_{\mathbb{R}^d} \mathrm{div}(Eu)\Phi'(u)m^p \;=\; \int_{\mathbb{R}^d} \mathrm{div}(E)(u\Phi'(u) - \Phi(u))m^p - \Phi(u)E\cdot\nabla m^p.$$

By definition of Φ, we obtain

$$\int_{\mathbb{R}^d} \mathrm{div}(Eu)u^{p-1}m^p \;=\; \int_{\mathbb{R}^d} |u|^p m^p \varphi_{m,p}, \tag{3.27}$$

where $\varphi_{m,p}$ is given by (3.8). Let now look at the term containing I. By using (3.15), we have

$$\int_{\mathbb{R}^d} I(u)u^{p-1}m^p \;=\; -\int_{\mathbb{R}^d} \mathfrak{D}_p(um) + \int_{\mathbb{R}^d} (m\, I(u) - I(mu))\,(um)^{p-1}.$$

By (3.26), when $|k| < \alpha \le 1$, we deduce the following inequality for I

$$\int_{\mathbb{R}^d} I(u)u^{p-1}m^p \;\le\; -\int_{\mathbb{R}^d} \mathfrak{D}_p(um) + C_k \int_{\mathbb{R}^d} |u|^p m^p \langle x\rangle^{k-\alpha}. \tag{3.28}$$

For $\alpha \in (0,2)$, when $kp \in (0, \alpha\wedge 1)$, we recall that by relation (3.16),

$$\mathfrak{D}_p(u) \;\simeq\; \frac{1}{p}\, I(|u|^p) - I(u)u^{p-1}.$$

Hence, using the fractional integration by parts formula (3.14), we get

$$\int_{\mathbb{R}^d} I(u)u^{p-1}m^p \;\le\; -C_p \int_{\mathbb{R}^d} \mathfrak{D}_p(u)m^p + \frac{1}{p}\int_{\mathbb{R}^d} |u|^p\, I(m^p).$$

By formula (3.19), it leads to

$$\int_{\mathbb{R}^d} I(u)u^{p-1}m^p + C_p \int_{\mathbb{R}^d} \mathfrak{D}_p(u)m^p \;\le\; C_k \int_{\mathbb{R}^d} |u|^p m^p \langle x\rangle^{-\alpha}. \tag{3.29}$$

Now we remark that, by relation (3.18)

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathfrak{D}_p(um) \;&\simeq\; \iint_{\mathbb{R}^{2d}} |(um)_*^{p/2} - (um)^{p/2}|^2 \,\mathrm{d}x_*\mathrm{d}x \\
&\lesssim\; 2\iint_{\mathbb{R}^{2d}} |u_*^{p/2} - u^{p/2}|^2 m_*^p + |m_*^{p/2} - m^{p/2}|^2 |u|^p \,\mathrm{d}x_*\mathrm{d}x \\
&\lesssim\; \int_{\mathbb{R}^d} \mathfrak{D}_p(u)m^p + \mathfrak{D}_p(m)|u|^p.
\end{aligned}$$

Moreover, since $\mathfrak{D}_p(m) \simeq \Gamma(m^{p/2}, m^{p/2})$, by the bound (3.14), we obtain

$$2\,\mathfrak{D}_p(m) \;\lesssim\; I(m^p) + m^{p/2}\left|I(m^{p/2})\right| \;\lesssim\; \frac{m^p}{\langle x\rangle^{\alpha}},$$

where we used (3.19) since $kp < \alpha$. Therefore, inequality (3.29) becomes

$$\int_{\mathbb{R}^d} I(u)u^{p-1}m^p + C_p \int_{\mathbb{R}^d} \mathfrak{D}_p(um) \;\le\; C_{k,p}\int_{\mathbb{R}^d} |u|^p m^p \langle x\rangle^{-\alpha}. \tag{3.30}$$

We conclude that (3.24) and (3.25) hold by combining the inequality for the part with E, equation (3.27) with the inequalities for the parts with I, (3.28) and (3.30). All these manipulation can be justified by taking $u_n = \chi_n(\rho_n * u) \to u$ where $\chi_n \in C_c^\infty$ is a cutoff function and $\rho_n \in C_c^\infty$ an approximation of δ_0. The main technical point is to obtain an estimate on the following commutator

$$r_n(u) \;:=\; (E\cdot\nabla u) * \rho_n - E\cdot\nabla(u*\rho_n).$$

In the spirit of DiPerna-Lions commutator estimate (see [78]) and Lemma 3.4, we obtain

$$r_n(u) \underset{n\to+\infty}{\longrightarrow} 0 \;\text{ in } L^p(m\,\langle x\rangle^{-\beta_+/p}),$$

which ends the proof. ☐

3.3 Well-posedness

This section is devoted to the proof of the part of Theorem 3.1 concerning existence and uniqueness of a continuous semigroup. In order to prove the existence of a solution to the (FFP) equation, we use a viscosity approximation of the equation and a truncation of E and I. We first prove the existence for the approximated problem in $L^2(M)$. We can identify the dual of $V := H^1(M) = \{u \in L^2(M), \nabla u \in L^2(M)\}$ to $H^{-1}(M)$ by defining $\langle f, g\rangle_{V',V} = \langle fM, gM\rangle_{H^{-1},H^1} = \int_{\mathbb{R}^d} fgM^2$. Moreover, $L^2(M)$ is a Hilbert space for the scalar product $\langle f, g\rangle_{L^2(M)} = \int_{\mathbb{R}^d} fgM^2$. Remark that in the case $\alpha > 1$, proving the existence is simpler as the divergence operator is bounded in H^α, so that we do not need to use a viscosity approximation.

Lemma 3.5 (Viscosity Approximation)**.** *Let $M := \langle x\rangle^k$ with $k \in \mathbb{R}$ and for $\varepsilon \in (0,1)$ define $\kappa_\alpha^\varepsilon(x) := \kappa_\alpha(x)\mathbb{1}_{\{\varepsilon<|x|<1/\varepsilon\}}$ and $I_\varepsilon(u) := \int_{\mathbb{R}^d} \kappa_{\alpha,*}^\varepsilon (u_* - u)\,\mathrm{d}x_*$. Then, there exists a unique solution in*

$$C^0([0,T], L^2(M)) \;\cap\; L^2((0,T), H^1(M)) \;\cap\; H^1((0,T), H^{-1}(M)),$$

to the problem

$$\partial_t f \;=\; \mathsf{L}_\varepsilon f \;:=\; \varepsilon\Delta f + I_\varepsilon(f) + \mathrm{div}(E_\varepsilon f), \tag{3.31}$$

with $f(0,\cdot) = f^{\mathrm{in}} \in L^2(M)$, $E_\varepsilon \in L^\infty$ and

$$\left(\mathrm{div}(E_\varepsilon) - E_\varepsilon\cdot\frac{\nabla M^2}{M^2}\right)_+ \in L^\infty.$$

Proof of Lemma 3.5. The result is an application of J.L.Lions Theorem (see for example [47, Théorème X.9]). We thus prove that the hypotheses of this theorem hold.

Step 1. Continuity of L_ε***.*** Let $(f,g) \in H^1(M)^2$. Then

$$\begin{aligned}\langle \mathsf{L}_\varepsilon f, g\rangle_{V',V} &= \int_{\mathbb{R}^d} -\varepsilon \nabla f \cdot \nabla(gM^2) + I_\varepsilon(f) g M^2 + \mathrm{div}(E_\varepsilon f) g M^2 \\ &= \int_{\mathbb{R}^d} \left(-\varepsilon \nabla f \cdot \nabla g - \varepsilon g \nabla f \cdot \frac{\nabla M^2}{M^2} + I_\varepsilon(f) g - E_\varepsilon f \left(\nabla g + g \frac{\nabla M^2}{M^2} \right) \right) M^2.\end{aligned}$$

Since $\kappa_\alpha^\varepsilon \in L^1$, we can write $I_\varepsilon(f) = \kappa_\alpha^\varepsilon * f - K_\varepsilon f$ where $K_\varepsilon = \|\kappa_\alpha^\varepsilon\|_{L^1}$. Using Peetre's inequality which tells that

$$\langle x+y \rangle \ \le\ \sqrt{2}\, \langle x \rangle \langle y \rangle\,,$$

and the fact that $\kappa_\alpha^\varepsilon$ is compactly supported, we get after a short computation

$$\left| \int_{\mathbb{R}^d} I_\varepsilon(f) g M^2 \right| \ \le\ C_\varepsilon K_\varepsilon \|f\|_{L^2(M)} \|g\|_{L^2(M)}. \tag{3.32}$$

Thus, using the Cauchy-Schwarz inequality, there exists $C_\varepsilon > 0$ such that

$$|\langle \mathsf{L}_\varepsilon f, g \rangle_{V',V}| \ \le\ \left(C_k(\varepsilon + \|E_\varepsilon\|_{L^\infty}) + C_\varepsilon K_\varepsilon \right) \|f\|_{H^1(M)} \|g\|_{H^1(M)},$$

where we used $|\nabla M^2| \le 2|k| M^2$. It proves that $\mathsf{L}_\varepsilon \in \mathscr{B}(V, V')$.

Step 2. For $f \in H^1(M)$, using (3.32) and the a priori estimate (3.27), we get

$$\begin{aligned}\langle \mathsf{L}_\varepsilon f, f\rangle_{V',V} &= \int_{\mathbb{R}^d} \left(I_\varepsilon(f) f - \varepsilon |\nabla f|^2 + f^2 \left(\mathrm{div}(E_\varepsilon) - E_\varepsilon \cdot \frac{\nabla M^2}{M^2} - \varepsilon \frac{\Delta M^2}{2M^2} \right) \right) M^2 \\ &\le -\varepsilon \|f\|^2_{H^1(M)} + C_{k,\varepsilon,E_\varepsilon,\kappa_\alpha^\varepsilon} \|f\|^2_{L^2(M)},\end{aligned}$$

where we used $|\nabla M^2| \le 2|k| M^2$ and $|\Delta M^2| \le 6|k| M^2$. Therefore, we can apply J.L.Lions Theorem. □

To get results in the good spaces, we will use the following injection that is a straightforward application of Hölder's inequality and the density of C_c^∞ in L^p.

Lemma 3.6. *Let* $(p,q) \in [1,+\infty]^2$ *and* $(l,k) \in \mathbb{R}^2$ *such that* $p \le q$ *and* $(l-k) > d\left(\frac{1}{p} - \frac{1}{q}\right)$. *Let* $M = \langle x \rangle^l$ *and* $m = \langle x \rangle^k$, *then*

$$L^q(M) \hookrightarrow L^p(m),$$

with dense and continuous embedding. In particular, if $l > k + \frac{d}{2}$ *and* $p \in [1,2]$, *we have the following embedding* $L^2(M) \hookrightarrow L^p(m)$.

We now can prove the existence of a weak solution by letting $\varepsilon \to 0$.

Lemma 3.7. *Let* $m = \langle x \rangle^k$ *with* $k \in (0, \alpha \wedge 1)$ *and* $p \in (1, p_\beta)$ *as defined by* (3.9) *(or* $p > 1$ *if* $\beta \le 0$*). Then there exists a unique weak solution* $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^p(m))$ *to the* (FFP) *equation.*

Proof of Lemma 3.7. We prove first existence of a solution in $L^p(m)$ by using the approximation in $L^2(M)$ and then we use it to prove existence in $L^1(m)$.

***Step 1. Existence in** $L^p(m)$ **for** $p > 1$.* Assume that $f^{\rm in} \in L^p(m)$ for $p \in (1,2]$. Then, by Lemma 3.6, there exists a family of functions $f^{\rm in}_\varepsilon \in L^2(M)$ such that

$$f^{\rm in}_\varepsilon \xrightarrow[\varepsilon\to 0]{L^p(m)} f^{\rm in}.$$

For a fixed $\varepsilon > 0$, let $\chi_\varepsilon \in C^\infty_c$ be a radial function such that $\chi_\varepsilon(x) = \tilde\chi_\varepsilon(|x|)$ where $\tilde\chi_\varepsilon$ is a decreasing function and $\mathbb{1}_{B(0,1/\varepsilon)} \le \chi_\varepsilon \le \mathbb{1}_{B(0,2/\varepsilon)}$. Let $f_\varepsilon \in C([0,T], L^2(M))$ be a solution of $\partial_t f_\varepsilon = \mathsf{L}_\varepsilon f_\varepsilon$ as given by Lemma 3.5, with $E_\varepsilon = E\chi_\varepsilon$. For such a E_ε, we have indeed $\mathrm{div}(E_\varepsilon) - E_\varepsilon \cdot \dfrac{\nabla M^2}{M^2}$ bounded above because of the fact that $E \in L^\infty_{\rm loc}$ and $\mathrm{div}(E)_+ \in L^\infty_{\rm loc}$.

Let $\rho \in \mathcal{D}(\mathbb{R}^{d+1}, \mathbb{R}_+)$ be such that $\int \rho = 1$ and $\mathrm{supp}(\rho) \subset (-1,0) \times B(0,1)$ so that $\rho_n(t,x) := n^{d+1}\rho(nt, n^d x)$ is an approximation of identity. The fractional Laplacian commutes with the convolution by smooth functions (which is an immediate property by using its Fourier definition (3.1)), thus the regularized function defined by $f_{\varepsilon,n} := f_\varepsilon * \rho_n \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d) \cap L^2(M)$ verifies in the classical sense the equation

$$\partial_t f_{\varepsilon,n} = \mathsf{L}_\varepsilon f_{\varepsilon,n} + r_n,$$

where

$$r_n = (E_\varepsilon \cdot \nabla f_\varepsilon) * \rho_n - E_\varepsilon \cdot \nabla f_{\varepsilon,n}.$$

As proved in [78, Lemma II.1], since $E_\varepsilon \in L^1((0,T), W^{1,r}_{\rm loc})$ for $r > 1$ such that $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$ and $f_\varepsilon \in L^\infty((0,T), L^2_{\rm loc})$, it holds

$$r_n \xrightarrow[n\to\infty]{} 0 \text{ in } L^1((0,T), L^p_{\rm loc}).$$

Moreover the convergence also holds in $L^1((0,T), L^p(m))$ because E_ε is compactly supported. Using inequality (3.24) or (3.25) for $I = I_\varepsilon$ and the fact that $\varphi_{m,p}$ is bounded from above, we obtain

$$\partial_t \left(\int_{\mathbb{R}^d} \frac{|f_{\varepsilon,n}|^p}{p} m^p \right) \le \int_{\mathbb{R}^d} |f_{\varepsilon,n}|^p m^p \left(C_k + \frac{\mathrm{div}(E_\varepsilon)}{q} - k \frac{E_\varepsilon \cdot x}{\langle x\rangle^2} \right) + |f_{\varepsilon,n}|^{p-1} |r_n| m^p.$$

For the part containing E_ε, we have

$$\frac{\mathrm{div}(E_\varepsilon)}{q} - k\frac{E_\varepsilon \cdot x}{\langle x\rangle^2} = \left(\frac{\mathrm{div}(E)}{q} - k \frac{E\cdot x}{\langle x\rangle^2} \right)\chi_\varepsilon + \frac{E \cdot \nabla(\chi_\varepsilon)}{q}.$$

By hypothesis, the first term is bounded above and the second term is negative since

$$E \cdot \nabla(\chi_\varepsilon) = E \cdot \frac{x}{|x|} \tilde\chi'(|x|) \le 0. \tag{3.33}$$

Using Hölder's inequality to control the error term, we obtain

$$\begin{aligned} \partial_t \left(\int_{\mathbb{R}^d} \frac{|f_{\varepsilon,n}|^p}{p} m^p \right) &\le C \int_{\mathbb{R}^d} |f_{\varepsilon,n}|^p m^p + \|r_n\|_{L^p(m)} \left(\int_{\mathbb{R}^d} |f_{\varepsilon,n}|^p m^p \right)^{1/p'} \\ &\le \left(C + \|r_n\|_{L^p(m)} \right) \int_{\mathbb{R}^d} |f_{\varepsilon,n}|^p m^p + \|r_n\|_{L^p(m)}, \end{aligned}$$

where we used the fact that for $\forall x \geq 0$, $x^{1/p'} \leq 1+x$ since $p' \in [2,\infty)$. Grönwall's inequality gives

$$\|f_{\varepsilon,n}\|^p_{L^p(m)} \;\leq\; e^{CT+p\|r_n\|_{L^1((0,T),L^p(m))}} \left(\|f^{\rm in}_{\varepsilon,n}\|^p_{L^p(m)} + p\|r_n\|_{L^1((0,T),L^p(m))} \right).$$

Passing to the limit in n, as $f_{\varepsilon,n} \to f_\varepsilon$ in $L^p(m)$, the error term cancels, hence

$$\|f_\varepsilon\|^p_{L^p(m)} \;\leq\; e^{CT}\|f^{\rm in}_\varepsilon\|^p_{L^p(m)}.$$

Thus, up to a subsequence, it converges in $\mathcal{D}'([0,T]\times\mathbb{R}^d)$ to $f \in L^\infty([0,T],L^p(m))$. Let $\varphi \in \mathcal{D}([0,T]\times\mathbb{R}^d)$. Then $\kappa^\varepsilon_{\alpha,*}\,|\varphi_* - \varphi| \leq \kappa_{\alpha,*}\,|\varphi_* - \varphi|$ which is integrable with respect to x_*, thus $I_\varepsilon(\varphi)$ converges to $I(\varphi)$ by the Lebesgue dominated convergence Theorem. Therefore, we have

$$\langle f_\varepsilon, I_\varepsilon(\varphi)\rangle_{\mathcal{D}',\mathcal{D}} \;\underset{\varepsilon\to 0}{\longrightarrow}\; \langle f, I(\varphi)\rangle.$$

It implies that $I_\varepsilon(f_\varepsilon) \underset{\varepsilon\to0}{\longrightarrow} I(f)$ in $\mathcal{D}'([0,T]\times\mathbb{R}^d)$. We can also easily check that $\mathrm{div}(E_\varepsilon f_\varepsilon) \underset{\varepsilon\to0}{\longrightarrow} \mathrm{div}(Ef)$ and $(\partial_t - \varepsilon\Delta)f_\varepsilon \underset{\varepsilon\to0}{\longrightarrow} \partial_t f$ in $\mathcal{D}'(\mathbb{R}_+\times\mathbb{R}^d)$. Therefore, we obtain the existence of $f \in L^\infty([0,T],L^p(m))$ verifying the (FFP) equation. Uniqueness follows directly by remarking that $f^{\rm in} = 0 \implies f = 0$.

Step 2. Existence in $L^1(m)$***.*** Consider now the case where $f^{\rm in} \in L^1(m)$. As $k < \alpha$, by Lemma 3.6 we can find $k < l < \alpha$ and $p \in (1,2)$ such that with $M = \langle x\rangle^l$, we have $L^p(M) \hookrightarrow L^1(m)$. Let $f^{\rm in}_n \underset{n\to\infty}{\longrightarrow} f^{\rm in}$ in $L^1(m)$ and f_n be the corresponding solution of the (FFP) given by the existence in the L^p case. Then, the same proof, but with the $L^1(m)$ estimates, gives

$$\|f_{n_1} - f_{n_2}\|_{L^1(m)} \;\leq\; e^{(C_0+C)T}\|f^{\rm in}_{n_1} - f^{\rm in}_{n_2}\|_{L^1(m)} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.34}$$

Therefore, f_n is a Cauchy sequence and we can again verify that it converges to a solution in $L^\infty((0,T),L^1(m))$ of the equation. □

Lemma 3.8. *Let $E \in L^\infty_{\rm loc}$, $m \in L^0(\mathbb{R},\mathbb{R}^*_+)$ and $f \in L^\infty((0,T),L^p(m))$ for $p \in (1,+\infty)$ be a weak solution of the (FFP) equation. Then we have the following continuity in time*

$$f \in C^0([0,T], w - L^p(m)),$$

where $w - L^p(m)$ indicates that we take the weak topology on $L^p(m)$.

Proof of Lemma 3.8. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$. As f is solution of (FFP) in $\mathcal{D}'((0,T)\times\mathbb{R}^d)$, taking $\psi\otimes\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ as test function, we can write

$$-\int_0^T\int_{\mathbb{R}^d} f(t,x)\partial_t\psi(t)\varphi(x)\,\mathrm{d}x\,\mathrm{d}t \;=\; \int_0^T\int_{\mathbb{R}^d} f(t,x)\psi(t)(I(\varphi) - E\cdot\nabla\varphi)(x)\,\mathrm{d}x\,\mathrm{d}t,$$

or equivalently

$$\partial_t u_\varphi = v_\varphi \text{ in } \mathcal{D}'(0,T),$$

with

$$u_\varphi : t \mapsto \int_{\mathbb{R}^d} f(t,\cdot)\varphi \qquad \text{and} \qquad v_\varphi : t \mapsto \int_{\mathbb{R}^d} f(t,\cdot)(I(\varphi) - E\cdot\nabla\varphi).$$

For $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and $E \in L^\infty_{\text{loc}}$, we have $I(\varphi) - E\cdot\nabla\varphi \in L^\infty(\langle x\rangle^{d+\alpha})$. Thus, as by Lemma 3.6, $f \in L^\infty((0,T), L^p(m)) \subset L^\infty((0,T), L^1(\langle x\rangle^{-(d+\alpha)}))$, we obtain that $u_\varphi \in L^\infty(0,T)$ and $v_\varphi \in L^\infty(0,T)$. Hence, $u_\varphi \in W^{1,\infty}(0,T) \subset C^0([0,T])$.

Let $p \neq 1$. We now show that the result is still true by replacing φ by $g \in L^{p'}(m^{-1})$. First, we remark that u_g is well defined in $L^\infty(0,T)$. Then, by the density of $\mathcal{D}(\mathbb{R}^d)$ in $L^{p'}$, there exists a sequence $(\tilde\varphi_n)_{n\in\mathbb{N}} \in \mathcal{D}(\mathbb{R}^d)^{\mathbb{N}}$ such that $\tilde\varphi_n \underset{n\to+\infty}{\longrightarrow} gm^{-1}$ in $L^{p'}$, or equivalently, there exists $\varphi_n := m\tilde\varphi_n \in \mathcal{D}(\mathbb{R}^d)$ such that $\varphi_n \underset{n\to+\infty}{\longrightarrow} g$ in $L^{p'}(m^{-1})$. We now look at the sequence of u_{φ_n} and write

$$\begin{aligned} \|u_{\varphi_n} - u_g\|_{C^0([0,T])} &= \left\| \int_{\mathbb{R}^d} f(t,\cdot)(\varphi_n - g) \right\|_{L^\infty(0,T)} \\ &\leq \|f\|_{L^\infty((0,T),L^p(m))} \|\varphi_n - g\|_{L^{p'}(m^{-1})} \underset{n\to+\infty}{\longrightarrow} 0. \end{aligned}$$

It proves that $u_g \in C^0(0,T)$. □

We can now combine the previous lemmas to give the proof of Theorem 3.1.

Proof of Theorem 3.1. Since the time continuity in the weak $\sigma(X, X')$ topology implies the continuity in the strong X topology (see e.g. [88]), we get the result in the case $p > 1$ by combining Lemma 3.7 and Lemma 3.8. If $p = 1$, we prove the time continuity differently. Using again an $L^p(M)$ approximation sequence f_n, we obtain from equation (3.34) that it is a Cauchy sequence in $C^0([0,T], L^1(m))$, since

$$\begin{aligned} \|f_{n_1} - f_{n_2}\|_{C^0([0,T],L^1(m))} &= \sup_{t\in(0,T)} \|f_{n_1} - f_{n_2}\|_{L^1(m)} \\ &\leq e^{(C_0+C)T} \|f^{\text{in}}_{n_1} - f^{\text{in}}_{n_2}\|_{L^1(m)} \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

from what we conclude that $f \in C^0([0,T], L^1(m))$. □

3.4 Additional properties for solutions to the equation

In this section, we prove that the semigroup associated to the (FFP) equation actually gives gains of regularity, integrability, weight and positivity, which is useful to retrieve quantitative estimates about the regularity of solutions, to prove uniform in time estimates in weighted Lebesgue spaces and existence and uniqueness of the steady state, as well as quantitative rate of decay towards equilibrium.

3.4.1 Gain of regularity and integrability

Proposition 3.9. *Let $f \in L^1(m)$ be a solution of the (FFP) equation as given in Theorem 3.1 for $m = \langle x \rangle^k$ with $k \in (0, \alpha \wedge 1)$. Then there exists $c > 0$ such that the following inequality holds*

$$\|f\|_{L^p(m)} \lesssim \left(c + \frac{d(p-1)}{\alpha t} \right)^{\frac{d}{q\alpha}} e^{t\lambda_1} \|f^{\text{in}}\|_{L^1(m)}, \tag{3.35}$$

where λ_1 is the growth bound of $e^{t\mathsf{L}}$ in $L^1(m)$, $q' = p \in [1, p_\beta)$ and if $\alpha \geq 1$, $p < \alpha/k$. Moreover, if $f^{\text{in}} \in L^p(m)$, we obtain the following Sobolev regularity

$$(fm)^{p/2} \in L^2((0,T), H^{\alpha/2}). \tag{3.36}$$

Remark 3.9. *Formula (3.35) can also be written in other words*

$$\|e^{t\mathsf{L}}\|_{L^1(m) \to L^p(m)} \lesssim \left(c + t^{\frac{-d}{\alpha q}} \right) e^{t\lambda_1}.$$

In order to show regularizing properties of the (FFP) equation, one possibility is to use a fractional variant of the Nash inequality in $L^p(m)$ spaces. This method goes back to the celebrated paper of Nash [173]. In the case of L^2 spaces, it is proved for example in [207, Lemma 5.2].

Lemma 3.10 (Fractional Nash inequality in $L^p(m)$)**.** *Let $p \in [1, 2]$ and $m = \langle x \rangle^k$ with $kp \in (0, \alpha \wedge 1)$ or $0 < k < \alpha < 1$. Then for any $u \in L^p(m)$, we have*

$$\int_{\mathbb{R}^d} I(u) u^{p-1} m^p \lesssim C_{k,p} \|u\|^p_{L^p(m)} - \left| (um)^{\frac{p}{2}} \right|^2_{H^{\alpha/2}} \tag{3.37}$$

$$\lesssim C_{k,p} \|u\|^p_{L^p(m)} - \|u\|^{p + \frac{q\alpha}{d}}_{L^p(m)} \|u\|^{\frac{-q\alpha}{d}}_{L^1(m)}. \tag{3.38}$$

Proof of Lemma 3.10. By the definition of the Sobolev seminorm (3.3) and the relation (3.18), we remark that

$$\int_{\mathbb{R}^d} \mathfrak{D}_p(w) \simeq |w^{\frac{p}{2}}|_{H^{\alpha/2}}.$$

Therefore, (3.37) is a consequence of inequalities (3.28) or (3.30). By using the following Gagliardo-Nirenberg inequalities (see for example [159])

$$\left\| (um)^{p/2} \right\|_{L^2} \lesssim \left| (um)^{p/2} \right|^{\theta}_{H^{\frac{\alpha}{2}}} \left\| (um)^{p/2} \right\|^{1-\theta}_{L^{2/p}},$$

with $\theta = \frac{p}{p + q\alpha/d}$, which can also be written

$$\|u\|^{p/\theta}_{L^p(m)} \lesssim \left| (um)^{p/2} \right|^2_{H^{\frac{\alpha}{2}}} \|u\|^{p(1/\theta - 1)}_{L^1(m)},$$

we deduce (3.38) from (3.37). □

Nash type inequalities let appear the following family of ordinary differential inequalities that can be solved explicitly and lead to the growth in time given by the following application of Gronwall's inequality.

Lemma 3.11. *Let $(A, B, C, b) \in \mathbb{R}^4$ and $y \in L^1_+(0,T)$ verifying in the weak sense $\partial_t X \leq BX - Ae^{-bCt}X^{1+C}$. Then, the following upper bound holds*

$$X \leq \frac{e^{-bt}}{A^{1/C}} \left((B-b) + \frac{1}{Ct} \right)^{1/C}.$$

We can now combine Lemma 3.11 with previous Nash type inequalities (3.37) and (3.38) to prove Proposition 3.9.

Proof of Proposition 3.9. Let $X = X(t) := \|f\|^p_{L^p(m)}$, $Y := \|f\|^p_{L^1(m)}$ and $\theta := \frac{\alpha}{d(p-1)} > 0$. The second fractional Nash inequality (3.38) can be written

$$\int_{\mathbb{R}^d} I(f) f^{p-1} m^p \;\leq\; \bar{C}X - \tilde{C}Y^{-\theta}X^{1+\theta}.$$

Thus, using the inequality (3.27) for the $\mathrm{div}(E\cdot)$ part of the operator L, we obtain

$$\begin{aligned} \partial_t X \;&=\; p\int_{\mathbb{R}^d} I(f) f^{q-1} m^p + p \int_{\mathbb{R}^d} f^p m^p \varphi_{m,p} \\ &\leq\; \left(p\bar{C} + C\right) X - p\tilde{C}Y^{-\theta}X^{1+\theta}. \end{aligned}$$

Using the fact that $Y \leq e^{q\lambda_1 t}Y(0)$ and Lemma 3.11, we obtain

$$X(t) \;\leq\; e^{q\lambda_1 t} \left(\frac{1}{p\tilde{C}}\right)^{\frac{1}{\theta}} \left(c_p + \frac{1}{\theta t}\right)^{\frac{1}{\theta}} Y(0),$$

with $c_p = p\bar{C} + C - q\lambda_1$. It proves (3.35). Let now $Z := |(fm)^{p/2}|_{H^{\alpha/2}}$ and assume $X(0)$ is bounded. Then by Theorem 3.1, we know that $X \leq e^{tp\lambda_p}X(0)$ for a given $\lambda_p \in \mathbb{R}$. Using now the first fractional Nash inequality (3.37), we have

$$\int_{\mathbb{R}^d} I(f) f^{p-1} m^p \;\leq\; \bar{C}X - Z^q.$$

It gives us, by integrating the a priori estimates with respect to time

$$\int_0^T Z^q \;\leq\; X(0) - X(T) + (p\bar{C} + C)\int_0^T X.$$

Therefore, we obtain

$$\int_0^T \|(fm)^{p/2}\|^q_{H^{\alpha/2}} \;\leq\; X(0)\left(1 + (p\bar{C} + C + 1)p\lambda_p e^{Tp\lambda_p}\right),$$

which gives (3.36). □

3.4.2 $L^1(m) \to L^\infty(m)$ Regularization when $\beta \le 0$

When $\beta \le 0$, we have a stronger regularization than Proposition 3.9 since the solutions are globally bounded in space. This property, which will hold also for the equilibrium, will be particularly useful to get the polynomial decay of Theorem 3.6.

Proposition 3.12. *Assume $\beta \le 0$. Let $f \in L^1(m)$ be a solution of the (FFP) equation as given in Theorem 3.1 with $m := \langle x \rangle^k$ with $2k \in (0, (\alpha \wedge 1))$ or $0 \le k < \alpha \le 1$. Then the following inequality holds*

$$\|f\|_{L^\infty(m)} \lesssim \left(C + t^{\frac{-d}{\alpha}}\right) e^{\frac{t}{2}(\lambda_1^* + \lambda_1)} \|f^{\text{in}}\|_{L^1(m)}, \tag{3.39}$$

where λ_1 is the growth bound of $e^{t\mathsf{L}}$ in $L^1(m)$, λ_1^* the growth bound of $e^{t\mathsf{L}^*}$ and $C \in \mathbb{R}$.

Remark 3.10. *Again, this technique goes back to the paper of Nash [173].*

Proof of Proposition 3.12. Since $\beta \le 0$, then Theorem 3.1 and the inequalities (3.24) or (3.25) hold in $L^p(m)$ for all $p \in [1, 2]$ and Proposition 3.9 holds for $p = 2$. It implies

$$\|e^{t\mathsf{L}}\|_{L^1(m) \to L^2(m)} \lesssim \left(C + t^{\frac{-d}{2\alpha}}\right) e^{t\lambda_1}. \tag{3.40}$$

Moreover, for g solution of the dual equation $\partial_t g = \mathsf{L}^* g := I(g) - E \cdot \nabla g$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} -(E \cdot \nabla g) g^{p-1} m^{-p} &= \frac{1}{p} \int_{\mathbb{R}^d} |g|^p \operatorname{div}(E m^{-p}) \\
&= \int_{\mathbb{R}^d} |g|^p m^{-p} \left(\frac{\operatorname{div}(E)}{p} - E \cdot \frac{\nabla m}{m} \right) \\
&\le \frac{\|\operatorname{div}(E)\|_{L^\infty}}{p} \int_{\mathbb{R}^d} |g|^p m^{-p},
\end{aligned}$$

by combining with formula (3.28) that still holds, we obtain the estimate

$$\partial_t \left(\int_{\mathbb{R}^d} |g|^p m^{-p} \right) = -\int_{\mathbb{R}^d} \mathfrak{D}_p(g m^{-1}) + \int_{\mathbb{R}^d} |g|^p m^{-p} \left(C_k + \frac{\|\operatorname{div}(E)\|_{L^\infty}}{p} \right).$$

Which is the equivalent of (3.24) for the dual equation in $L^p(m^{-1})$. With the same proof, we get that Theorem 3.1 and Proposition 3.9 also hold in $L^p(m^{-1})$ for $p \in [1, 2]$, from what we deduce

$$\|e^{t\mathsf{L}^*}\|_{L^1(m^{-1}) \to L^2(m^{-1})} \lesssim \left(c + t^{\frac{-d}{2\alpha}}\right) e^{t\lambda_1^*}, \tag{3.41}$$

where λ_1^* is the growth bound of $e^{t\mathsf{L}^*}$ in $L^1(m^{-1})$. Since the dual of $L^1(m^{-1})$ and $L^2(m^{-1})$ can be identified with $L^\infty(m)$ and $L^2(m)$, we deduce from (3.41) that

$$\|e^{t\mathsf{L}}\|_{L^2(m) \to L^\infty(m)} \lesssim \left(c + t^{\frac{-d}{2\alpha}}\right) e^{t\lambda_1^*}.$$

And combining with (3.40), by writing $e^{t\mathsf{L}} = e^{\frac{t}{2}\mathsf{L}} e^{\frac{t}{2}\mathsf{L}}$, we end up with

$$\|e^{t\mathsf{L}}\|_{L^1(m) \to L^\infty(m)} \lesssim \left(C + t^{\frac{-d}{\alpha}}\right) e^{\frac{t}{2}(\lambda_1^* + \lambda_1)},$$

which ends the proof. $\square$

3.4.3 Gain of positivity

We prove in this section the gain and the propagation of strict positivity. It will be useful to prove the uniqueness of the steady state and also, as explained in Proposition 3.24, to get asymptotic estimates when we are not able to prove that the steady state is bounded and use the Poincaré inequality. The first proposition is the classical maximum principle.

Proposition 3.13 (Weak Parabolic Maximum Principle)**.** *Assume that the conditions of Proposition 3.3 are satisfied and let $f \in L^p(\mathbb{R}_+, L^p(m \langle x \rangle^{\beta_+/p}))$ be such that*

- $(\partial_t - \mathsf{L}) f \geq 0$,
- $f(0, \cdot) = f^{\text{in}} \geq 0$.

Then $f \geq 0$.

Proof of Proposition 3.13. Let $g \in L^p(m \langle x \rangle^{\beta_+/p})$, $g_- := (-g)_+$ its negative part and $\Phi(g) := g_+^p$. We remark that

$$\int_{\mathbb{R}^d} I(g)\, \Phi'(g) m^p \;\leq\; p \int_{\mathbb{R}^d} I(g_+)\, g_+^{p-1} m^p,$$

because, as $(g_-)(g_+) = 0$, we have

$$\begin{aligned} - \int_{\mathbb{R}^d} I(g_-) g_+^{p-1} m^p \;&=\; - \iint_{\mathbb{R}^{2d}} \kappa_{\alpha,*} ((g_-)_* - g_-) g_+^{p-1} m^p \, \mathrm{d}x_* \mathrm{d}x \\ &=\; - \iint_{\mathbb{R}^{2d}} \kappa_{\alpha,*} (g_-)_* (g_+^{p-1}) m^p \, \mathrm{d}x_* \mathrm{d}x \;\leq\; 0. \end{aligned}$$

Thus, if g is such that $\partial_t g \leq \mathsf{L} g$, we get

$$\partial_t \left(\int_{\mathbb{R}^d} |g_+|^p m^p \right) \;\leq\; p \int_{\mathbb{R}^d} (\mathsf{L} g) g_+^{p-1} m^p \;\leq\; p \int_{\mathbb{R}^d} \mathsf{L}(g_+) g_+^{p-1} m^p.$$

Using the a priori estimates (3.24) or (3.25), we obtain

$$\int_{\mathbb{R}^d} |g_+|^p m^p \;\leq\; e^{\lambda t} \int_{\mathbb{R}^d} |g_+^{\text{in}}|^p m^p.$$

We conclude by taking $f = -g$ and remarking that $f_-^{\text{in}} = 0 \implies f_- = 0$. □

The second proposition claims that the solutions to the (FFP) equations are actually bounded by below by a strictly positive function as soon as they have positive mass in a compact set. It implies in particular the strong maximum principle.

Proposition 3.14. *Let f be a solution to the (FFP) equation with initial condition $f^{\text{in}} \in L^1_+ \cap L^p(m)$. Then for any $\gamma > \alpha + \beta_+$ and $R > 0$, there exists $C > 0$ and $\lambda > 0$ such that for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$*

$$f(t, x) \;\geq\; C \, \frac{t e^{-\lambda t}}{\langle x \rangle^{d+\gamma}} \int_{B_R} f^{\text{in}},$$

where B_R denotes the ball of size R.

For a given $r > 0$, we define $\chi := \mathbb{1}_{B_r}$, $\chi^c := 1 - \chi$, $\kappa^c := \kappa_\alpha \chi^c + \kappa_\alpha(r)\chi = \min(\kappa_\alpha, \kappa_\alpha(r))$ and $\kappa := \kappa_\alpha - \kappa^c \geq 0$. As $\kappa^c \in L^1$, we will denote by $K^c := \|\kappa^c\|_{L^1}$ and will decompose I into

$$\begin{aligned} I_c(u) &:= \int_{\mathbb{R}^d} \kappa^c_* \,(u_* - u)\,\mathrm{d}x_* \;=\; \kappa^c * u - K^c u \\ I_\chi(u) &:= \int_{\mathbb{R}^d} \kappa_* \,(u_* - u)\,\mathrm{d}x_* \;=\; \int_{|x-y|<r} \kappa(x-y)\,(u(y) - u(x))\,\mathrm{d}y. \end{aligned}$$

Then we define the splitting

$$\mathsf{L} = \mathsf{A} + \mathsf{B},$$

where

$$\mathsf{A}u = \kappa^c * u \text{ and } \mathsf{B} = (I_\chi + \mathrm{div}(E\ \cdot) - K^c).$$

Since the second operator still generates a positive semigroup, the strategy is to use the following Duhamel's formula (see e.g. [9])

$$e^{t\mathsf{L}} \;=\; e^{t\mathsf{B}} + e^{t\mathsf{L}} \star \mathsf{A} e^{t\mathsf{B}},$$

where we defined the time convolution of two operators by

$$\mathsf{U} \star \mathsf{V} : t \mapsto \int_0^t \mathsf{U}(t-s)\mathsf{V}(s)\,\mathrm{d}s,$$

and to prove that A gives a gain of positivity while $e^{t\mathsf{L}}$ propagates the lower bound. These properties are given in the following lemmas. We will need the following bound by below

Lemma 3.15 (Bound by below for $I(m)$)**.** *Let $m(x) := \langle x \rangle^k$ with $k < \alpha$. Then*

$$I(m) \;\geq\; C_k \langle x \rangle^{-(d+\alpha)} - \tilde{C}_k m.$$

Proof of Lemma 3.15. We use the above splitting of the fractional Laplacian into $I = I_\chi + I_c$ for $\chi = \mathbb{1}_{B_1}$.We first deal with $I_\chi(m)$ and remark that

$$I_\chi(u) \;:=\; \int_{\mathbb{R}^d} \kappa_* \,(u_* - u - (x_* - x)\cdot \nabla u)\,\mathrm{d}x_*. \tag{3.42}$$

By a second order Taylor approximation, for $z \in B_1$, we obtain

$$|m(x+z) - m(x) - z \cdot \nabla m(x)| \;\leq\; \frac{|z|^2}{2} \left\| \nabla^2 m \right\|_{L^\infty(B_1(x))}.$$

Thus, by the change of variable $z = x - x_*$ in (3.42), we can write

$$\begin{aligned} |\, I_\chi(m)| &\leq \frac{1}{2} \left\| \nabla^2 m \right\|_{L^\infty(B_1(x))} \int_{|z|<1} \frac{\chi(z)}{|z|^{d+\alpha-2}}\,\mathrm{d}z \\ &\leq \frac{\omega_d}{2(2-\alpha)} \|\chi\|_{L^\infty} \left\| \nabla^2 m \right\|_{L^\infty(B_1(x))}. \end{aligned}$$

In particular, since $m = \langle x\rangle^k$, we have

$$\left\|\nabla^2 m\right\|_{L^\infty(B_1(x))} \;\le\; \sup_{|z|<R} |k(|k|+3)\,\langle x+z\rangle^{k-2}\,|.$$

Peetre's inequality tells that for all $(x,z) \in \mathbb{R}^{2d}$, we have

$$\langle x+z\rangle^{k-2} \;\le\; \sqrt{2}^{|k-2|}\,\langle x\rangle^{k-2}\,\langle z\rangle^{|k-2|}\,.$$

Since $\langle z\rangle \le \langle 1\rangle$, we obtain

$$|I_\chi(m)| \;\le\; C_k\,\langle x\rangle^{k-2}\,. \tag{3.43}$$

Now deal with the second part. Since $I_c(m) = \kappa^c * m - K^c m$, we just have to remark that

$$\kappa^c * m \;\ge\; \kappa^c * (m(1)\,\mathbb{1}_{B_1}) \;\ge\; \frac{C}{(|x|+1)^{d+\alpha}}.$$

Then, by combining with (3.43), we obtain

$$I(m) \;\ge\; \frac{C}{(|x|+1)^{d+\alpha}} - \left(K^c + \frac{C_k}{\langle x\rangle^2}\right) m.$$

what gives the result. □

Lemma 3.16 (Propagation of positivity)**.** *Let $f \in L^p((0,T), L^p(m\,\langle x\rangle^{\beta_+/p}))$ be a solution to the (FFP) equation such that $f^{\rm in} > \frac{1}{\langle x\rangle^{d+\gamma}}$ with $\gamma > \alpha+\beta$. Then there exists $\lambda > 0$ such that*

$$f(t,x) \;\ge\; \frac{e^{-\lambda t}}{\langle x\rangle^{d+\gamma}}. \tag{3.44}$$

Proof of Lemma 3.16*.* We prove that for λ large enough, $g(t,x) := \mathfrak{m}(x)\psi(t)$ with $\psi(t) = e^{-\lambda t}$ and $\mathfrak{m}(x) = \langle x\rangle^k$ with $k < -(d+\alpha+\beta)$ is a subsolution. By Lemma 3.15, we have, indeed

$$I(\mathfrak{m}) \;\ge\; \left(C_k\,\langle x\rangle^{-(d+\alpha+k)} + \tilde{C}_k\right)\mathfrak{m}.$$

We deduce

$$\begin{aligned}
(\partial_t - \mathsf{L})g \;&=\; -\lambda g - I(\mathfrak{m})\psi(t) - \operatorname{div}(E\mathfrak{m})\psi(t)\\
&\le\; \left(-\lambda - C_k\,\langle x\rangle^{-(d+\alpha+k)} + \tilde{C}_k - \operatorname{div}(E) - E\frac{\nabla\mathfrak{m}}{\mathfrak{m}}\right) g\\
&\le\; \left(\tilde{C}_k - \lambda - C_k\,\langle x\rangle^{\beta+\varepsilon} + C\,\langle x\rangle^{\beta}\right) g,
\end{aligned}$$

where $\varepsilon := -(k+d+\alpha+\beta) > 0$. Therefore, by taking λ sufficiently large we obtain $(\partial_t - \mathsf{L})g \le 0$, i.e. g is a subsolution to the equation. As $g \in L^p_{t,x}(\langle x\rangle^{\alpha+\beta_+/p})$, we can apply the weak parabolic maximum principle, Proposition 3.13, to $f-g$ and we get that $f \ge g$. □

Lemma 3.17 (Creation of positivity)**.** *For $u \in L^1_+$ the following lower bound holds*

$$\kappa^c * u \;\geq\; \frac{C}{\langle x\rangle^{d+\alpha}} \int_{B_R} u, \tag{3.45}$$

where $C = (\sqrt{2}\,\max(r, R, 1))^{-(d+\alpha)}$.

Proof of Lemma 3.17. If $y \in B_R$, then $|x-y| \leq |x| + R$. We deduce the following lower bound

$$\kappa^c * u(x) \;\geq\; \int_{|y|<R} \mathbb{1}_{|x-y|<r} \frac{u(y)}{|r|^{d+\alpha}} + \mathbb{1}_{|x-y|>r} \frac{u(y)}{(|x|+R)^{d+\alpha}}\,\mathrm{d}y.$$

Let $r_1 := \max(r, R, 1)$. As $|x| + R \leq |x| + r_1$ and $r \leq |x| + r_1$, we get

$$\kappa^c * u(x) \;\geq\; \frac{1}{(|x|+r_1)^{d+\alpha}} \int_{|y|<R} u(y)\,\mathrm{d}y \;\geq\; \frac{C}{\langle x\rangle^{d+\alpha}} \int_{B_R} u,$$

where $C = (\sqrt{2}\,r_1)^{-(d+\alpha)}$. □

Now we prove that $e^{t\mathsf{B}}$ propagates the fact to have a positive mass in a compact set.

Lemma 3.18. *Let $u \in L^1(m)$ and $R > 0$. Then for all $\delta > 0$, there exists $\lambda_\delta > 0$ such that*

$$\int_{B_{R+\delta}} e^{t\mathsf{B}} u \;\geq\; e^{-\lambda_\delta t} \int_{B_R} u. \tag{3.46}$$

Proof of Lemma 3.18. Let $\eta_0 \in C^\infty_c$ be a radially decreasing function such that $\mathbb{1}_{B_{\bar{R}}} \leq \eta_0 \leq \mathbb{1}_{B_R}$ and $\eta_0 > 0$ on B_R. We also define for all $t > 0$, $\eta_t := e^{-\lambda t}\eta_0$ for a given $\lambda > 0$. By construction, this is a subsolution of $\partial_t + E\cdot\nabla$ since

$$\partial_t \eta + E \cdot \nabla\eta \;=\; -\lambda\eta - \left(E \cdot \frac{x}{|x|}\right)|\nabla\eta| \;\leq\; -\lambda\eta.$$

Our goal is to prove that for λ sufficiently large, we even better have $\partial_t\eta + E\cdot\nabla\eta - I_\chi(\eta) \leq 0$. Therefore, we look at the behaviour of $I_\chi(\eta)$ where $\chi = \mathbb{1}_{B_r}$. For $|x| > R$ we have

$$I_\chi(\eta) \;=\; \int_{|x-y|<r} \frac{\eta(y)}{|x-y|^{d+\alpha}}\,\mathrm{d}y \;\geq\; \frac{1}{r^{d+\alpha}} \int_{B_r(x)} \eta \;\geq\; 0,$$

where $B_r(x)$ is the ball of center x and radius r. In particular, defining $j_R := I_\chi(\eta)(x)$ for $|x| = R$, we have $j_R > 0$. As $\eta \in C^\infty$, we easily deduce $I_\chi(\eta) \in C^\infty$ and the existence of $R' \in (\bar{R}, R)$ such that for all $|x| \in [R', R]$, $I_\chi(\eta) \geq j_R/2 > 0$. Therefore, we obtain the following cases

$$\begin{aligned} |x| > R' \;&\Longrightarrow\; I_\chi(\eta) + \lambda\eta \;\geq\; \lambda\eta \;\geq\; 0 \\ |x| < R' \;&\Longrightarrow\; I_\chi(\eta) + \lambda\eta \;\geq\; \lambda\eta(R') - \|I_\chi(\eta)\|_{L^\infty}, \end{aligned}$$

and the latter is positive for λ sufficiently large. As $\eta \in C^\infty([0,T]\times B_R)$ all the estimates can easily be made uniform in time and we therefore obtain that

$$(\partial_t - \mathsf{B}^*)\eta \;\le\; 0.$$

In particular, by application of the maximum principle (Proposition 3.13) we obtain that $e^{t\mathsf{B}^*}\mathbb{1}_{B_R} \ge e^{t\mathsf{B}^*}\eta_0 \ge \eta \ge e^{-\lambda t}\mathbb{1}_{B_{\bar R}}$. By the dual definition of positivity, we obtain inequality (3.46). □

We can now prove the gain of positivity for the (FFP) equation.

Proof of Proposition 3.14. We combine (3.45) and (3.46) to get

$$\mathsf{A}\, e^{s\mathsf{B}} f^{\mathrm{in}} \;\ge\; \frac{C_{R,\delta,\chi} e^{-\lambda_\delta s}}{\langle x\rangle^{d+\alpha}} \int_{B_R} f^{\mathrm{in}},$$

where $C = (2\sqrt{2})^{-(d+\alpha)}$. By propagation of the positivity (Lemma 3.16), for any $\gamma > \alpha + \beta_+$,

$$e^{(t-s)\mathsf{L}}\mathsf{A}\, e^{s\mathsf{B}} f^{\mathrm{in}} \;\ge\; \frac{C_{R,\delta,\chi} e^{-\lambda(t-s)} e^{-\lambda_\delta s}}{\langle x\rangle^{d+\gamma}} \int_{B_R} f^{\mathrm{in}} \ge \frac{C_{R,\delta,\chi} e^{-\lambda t}}{\langle x\rangle^{d+\gamma}} \int_{B_R} f^{\mathrm{in}},$$

since up to taking the maximum of λ and λ_δ, one can assume that $\lambda = \lambda_\delta$. In conclusion, we obtain the claimed result by integrating on $s \in [0,t]$ and using the fact that $e^{t\mathsf{B}} \ge 0$ and that by Duhamel's formula

$$e^{t\mathsf{L}} \;=\; e^{t\mathsf{B}} + e^{t\mathsf{L}}\mathsf{A} \star e^{t\mathsf{B}} \;\ge\; e^{t\mathsf{L}}\mathsf{A} \star e^{t\mathsf{B}}.$$

□

3.5 Existence and uniqueness of the steady state

3.5.1 Splitting of L as a bounded and a dissipative part

Following the ideas of [167, 130], this section uses a splitting of the operator L in a dissipative part in $\mathsf{B} \in \mathscr{B}(L^1(m), L^p(m^\theta))$ and a bounded part $\mathsf{A} \in \mathscr{B}(L^1, L^1(m))$ in order to bound uniformly in time the solution to the (FFP) equation and obtain the existence of a steady state. We define the new splitting as $\mathsf{L} = \mathsf{A} + \mathsf{B}$ with

$$\mathsf{A} := c\chi_R \text{ and } \mathsf{B} := \mathsf{L} - c\chi_R,$$

where $c > 0$ is a large enough constant and $\mathbb{1}_{B_R} \le \chi_R \le \mathbb{1}_{B_{2R}}$ is a smooth cutoff function.

Proposition 3.19. *Assume $\beta > -\alpha$ and let $k \in (0, \alpha \wedge 1)$ and $p \in (1, p_\beta)$. Then there exists $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|e^{t\mathsf{B}}\|_{\mathscr{B}(L^p(m), L^p(m^\theta))} \;\lesssim\; \omega(t), \tag{3.47}$$

where

- *if $\beta \geq 0$, then $\theta = 1$ and $\omega(t) = e^{-bt}$,*
- *if $\beta \in (-\alpha, 0)$, then θ is any number in $(0, 1]$, $\omega(t) = \langle t \rangle^{-k(1-\theta)/|\beta|}$ and we require $p < \alpha/k$ if $k > \alpha + \beta$.*

In particular, if $\beta > -\alpha$ and $p_\beta \geq 1$, there exists (p, θ) such that $\omega \in L^1(\mathbb{R}_+)$. Moreover, the gain of integrability also holds for B and writes

$$\|e^{t\mathsf{B}} f\|_{\mathscr{B}(L^1(m), L^p(m))} \lesssim t^{-\frac{d}{q\alpha}}, \tag{3.48}$$

where we recall that $q = p'$.

Proof of Proposition 3.19. By inequality (3.24), if $0 < kp < \alpha \wedge 1$, we have

$$\frac{1}{p} \partial_t \left(\int_{\mathbb{R}^d} |f|^p m^p \right) \leq \frac{1}{p} \int_{\mathbb{R}^d} |f|^p m^p \left(\frac{C_k}{\langle x \rangle^\alpha} + \varphi_{m,p} - c\chi_R \right) - C \int_{\mathbb{R}^d} \mathfrak{D}_p(fm).$$

Or, by inequality (3.25), we can also get for $k \in (0, \alpha)$,

$$\frac{1}{p} \partial_t \left(\int_{\mathbb{R}^d} |f|^p m^p \right) \leq \frac{1}{p} \int_{\mathbb{R}^d} |f|^p m^p \left(\frac{C_k}{\langle x \rangle^{\alpha - k}} + \varphi_{m,p} - c\chi_R \right) - C \int_{\mathbb{R}^d} \mathfrak{D}_p(fm).$$

From (3.9), for $p < p_\beta$, we have $\varphi_{m,p} \leq b \mathbb{1}_\Omega - a \langle x \rangle^\beta$. Therefore, since $\beta > -\alpha$, if $kp < \alpha$ or $k < \alpha + \beta$, for c and R large enough, we obtain

$$\frac{1}{p} \partial_t \left(\int_{\mathbb{R}^d} |f|^p m^p \right) \leq -a \int_{\mathbb{R}^d} |f|^p \langle x \rangle^{kp+\beta} - C \int_{\mathbb{R}^d} \mathfrak{D}_p(fm), \tag{3.49}$$

with $a > 0$. In particular

$$\|e^{t\mathsf{B}}\|_{\mathscr{B}(L^p(m))} \leq 1, \tag{3.50}$$

which proves inequality (3.47) for small times. If $\beta \geq 0$, since $m^p \leq \langle x \rangle^{kp+\beta}$, the result immediately follows be Grönwall's inequality. Assume now $\beta < 0$ and let $\varepsilon := pk(1-\theta) > 0$. By Hölder's inequality, we have

$$\int_{\mathbb{R}^d} |f|^p m^{\theta p} \leq \left(\int_{\mathbb{R}^d} |f|^p \langle x \rangle^{\theta kp+\beta} \right)^{\varepsilon/(|\beta|+\varepsilon)} \left(\int_{\mathbb{R}^d} |f|^p m^p \right)^{|\beta|/(|\beta|+\varepsilon)}.$$

Combining it with (3.49) (where we replace k by θk) and (3.50) leads to

$$\partial_t \left(\int_{\mathbb{R}^d} |f|^p m^{\theta p} \right) \leq -\bar{a} \left(\int_{\mathbb{R}^d} |f|^p m^{\theta p} \right)^{1+|\beta|/\varepsilon} \left(\int_{\mathbb{R}^d} |f^{\text{in}}|^p m^p \right)^{-|\beta|/\varepsilon}. \tag{3.51}$$

By Grönwall's inequality, we obtain

$$\int_{\mathbb{R}^d} |f|^p m^{\theta p} \lesssim \frac{1}{t^{\varepsilon/|\beta|}} \int_{\mathbb{R}^d} |f^{\text{in}}|^p m^p.$$

It proves inequality (3.47) for large times. Moreover, using this time the second term of the right-hand side in (3.49) and following the same proof as in Proposition 3.9, we get

$$\|e^{t\mathsf{B}} f\|_{\mathscr{B}(L^1(m), L^p(m))} \lesssim t^{-\frac{d}{q\alpha}} e^{t\lambda_1}.$$

But using (3.50) for $p = 1$ proves that we can take $\lambda_1 = 1$. It concludes the proof. □

3.5.2 Existence of a unique steady state

With this dissipative estimate, the gain of integrability property of Proposition 3.9 and the properties of A, we obtain the following global in time estimates.

Proposition 3.20 (Global propagation of L^p norms). *Assume $\beta > -\alpha$ and let f be a solution of the (FFP) under the assumptions of Theorem 3.1 with $f^{\text{in}} \in L^1 \cap L^p(m^\theta)$. Then, if $\beta \geq 2$ and $p < p_\beta$ or if $\beta \in (-\alpha, 0)$ and (p,k) is such that $q = p' > d/\alpha$ and there exists $\theta \in (0,1)$ such that Proposition 3.19 holds with $\omega \in L^1(\mathbb{R}_+)$, there exists $C > 0$ such that*

$$\|e^{t\mathsf{L}} f^{\text{in}}\|_{L^p(m^\theta)} \ \leq\ C \left(\|f^{\text{in}}\|_{L^1} + \|f^{\text{in}}\|_{L^p(m^\theta)} \right).$$

Proof of Proposition 3.20. By noticing that $\mathsf{A} \in \mathscr{B}(L^1, L^1(m))$, thanks to Proposition 3.19, we obtain the following sequence of estimates

$$L^1 \xrightarrow[e^{t\mathsf{L}}]{1} L^1 \xrightarrow[\mathsf{A}]{\|\mathsf{A}\|} L^1(m) \xrightarrow[e^{t\mathsf{B}/2}]{\omega_2(t)} L^p(m) \xrightarrow[e^{t\mathsf{B}/2}]{\omega(t)} L^p(m^\theta),$$

where $\omega_2(t) = t^{-d/q\alpha}$ (which is integrable in 0 since $q > d/\alpha$) and we have indicated the linear operator under the arrow and the corresponding growth rate above the arrows. Hence, by remarking that $\omega\omega_2 \in L^1(\mathbb{R}_+)$ using the following Duhamel's Formula

$$e^{t\mathsf{L}} = e^{t\mathsf{B}} + e^{t\mathsf{B}/2} e^{t\mathsf{B}/2} \star \mathsf{A} e^{t\mathsf{L}},$$

and the global boundedness of $e^{t\mathsf{B}}$ in $L^p(m^\theta)$ given by (3.50), we deduce the announced result. $\square$

This proposition together with the positivity properties of the semigroup are sufficient to prove existence and uniqueness of the steady state.

Proof of Theorem 3.5. Since we have obtained a bound, uniform in time, in the weakly sequentially compact set $L^1_+ \cap L^p(m)$, a fixed-point argument allows us to claim the existence of a stationary state. Following the same proof as in [170, Lemma 3.6] or [130, Theorem 5.1], we obtain from the previous estimates the existence a stationary state $F \in L^1 \cap L^p(m)$ to the (FFP) equation.

Moreover, by the positivity results obtained in Proposition 3.14 and since $1 \in L^{p'}(m^{-1}) \cap L^\infty$ for $p < \dfrac{d}{d-k}$, we obtained the following facts

- There exists $F \in L^p(m) \cap L^1_+$ such that $\mathsf{L}F = 0$,
- $\mathsf{L}^* 1 = 0$ and $1 \in (L^p(m) \cap L^1)'_+$,
- L satisfies the strong and the weak maximum principle.

As a consequence of the Krein-Rutman Theorem (see e.g. [168, Theorem 5.3]), we deduce the uniqueness of a stationary state $F \in L^p(m) \cap L^1_+$ of given mass $\|F\|_{L^1} = \|f^{\text{in}}\|_{L^1}$. It finishes the proof of Theorem 3.5. $\square$

3.6 Polynomial Convergence to the equilibrium for $\beta \in (-\alpha, 0)$

When $\beta \in (-\alpha, 0)$, the force field seems not confining enough to get exponential convergence since the derivatives of weighted Lebesgue norms let appear Lebesgue norms with smaller weights. Moreover, when $\beta < 2 - \alpha$, the effect of the force field at infinity is dominated by the effect of the fractional Laplacian, which prevent us from proving any explicit convergence result with our method.

3.6.1 Generalized relative entropy

In this section, we make a remark about the fact that we can already easily prove a non-quantitative version of the convergence toward equilibrium by generalized entropy method. Assume that there exists a steady state $F > 0$ to the (FFP) equation and let f be a solution of the equation of mass 0. Then for $h := f/F$, by integration by parts, the following computation formally holds

$$\frac{1}{p}\partial_t \left(\int_{\mathbb{R}^d} |h|^p F \right) \;=\; \int_{\mathbb{R}^d} (I(h^{p-1})h - hE \cdot \nabla h^{p-1})F.$$

Then, since by formula (3.17)

$$\mathfrak{D}_p(h) \;\simeq\; \frac{1}{q}\, I(|h|^p) - h\, I(h^{p-1}),$$

we get

$$\begin{aligned}
\frac{1}{p}\partial_t \left(\int_{\mathbb{R}^d} |h|^p F \right) \;&\leq\; \frac{1}{q} \int_{\mathbb{R}^d} (I(|h|^p) - p h^{p-1} E \cdot \nabla h)F - C_p \int_{\mathbb{R}^d} \mathfrak{D}_p(h) F \\
&\leq\; \frac{1}{q} \int_{\mathbb{R}^d} (I(|h|^p) - E \cdot \nabla |h|^p)F - C_p \int_{\mathbb{R}^d} \mathfrak{D}_p(h) F \\
&\leq\; \frac{1}{q} \int_{\mathbb{R}^d} |h|^p (I(F) + \operatorname{div}(EF)) - C_p \int_{\mathbb{R}^d} \mathfrak{D}_p(h) F \\
&\leq\; -C_p \int_{\mathbb{R}^d} \mathfrak{D}_p(h) F.
\end{aligned}$$

Thus, we obtain

$$\frac{1}{p}\partial_t \left(\int_{\mathbb{R}^d} |f|^p F^{1-p} \right) \;\leq\; -C_p \int_{\mathbb{R}^d} \mathfrak{D}_p \left(\frac{f}{F} \right) F. \tag{3.52}$$

Since $\mathfrak{D}_p(h) \geq 0$ and $\mathfrak{D}_p(h) = 0 \Leftrightarrow h$ is constant $\Leftrightarrow f = F$ (by conservation of the mass), it implies that $\int_{\mathbb{R}^d} |h|^p F$ is a strict Lyapunov functional, which implies the convergence to the equilibrium in $L^p(F^{-1/q})$ (see for example [164, Chapter 5] or [114]). However, we will prove that with other techniques we will get an explicit rate of convergence.

3.6.2 Fractional Poincaré-Wirtinger inequality

We prove in this section an inequality looking like a fractional Poincaré-Wirtinger inequality on a bounded set Ω, but for the p-dissipation $\mathfrak{D}_p$ instead of a fractional gradient and for functions such that the mass is zero on the whole space (i.e. u such that $\langle u \rangle_\mu = 0$).

We define the diameter of Ω as $\operatorname{diam}(\Omega) := \sup_{(x,y)\in\Omega^2}(|x-y|)$. Moreover, we introduce the following notation for the mass and the L^p norm of a function u for a measure μ,

$$\langle u \rangle_{\mu,\Omega} \;:=\; \frac{1}{\mu(\Omega)}\int_\Omega u\mu, \qquad\qquad \|u\|^p_{L^p_\mu(\Omega)} \;:=\; \int_\Omega |u|^p \mu,$$

and we will use the shortcuts $\langle u \rangle_\mu := \langle u \rangle_{\mu,\mathbb{R}^d}$ and $\|u\|^p_{L^p_\mu} = \|u\|^p_{L^p_\mu(\mathbb{R}^d)}$.

Proposition 3.21. *Let $\mu \in L^\infty_{\mathrm{loc}} \cap L^1_+$. Then for all $u \in L^p_\mu$ such that $\langle u \rangle_\mu = 0$, for all $\Omega \subset \mathbb{R}^d$ bounded, the following inequality holds*

$$\int_\Omega |u|^p \mu \;\le\; C_{\mathrm{PW}} \int_\Omega \mathfrak{D}_p(u)\mu + \varepsilon_\Omega \|u\|^{p-1}_{L^p_\mu(\Omega)} \|u\|_{L^p_\mu(\Omega^c)}\,,$$

where $C_{\mathrm{PW}} = \operatorname{diam}(\Omega)^{d+\alpha} \|\mu\|_{L^\infty(\Omega)}$ and $\varepsilon_\Omega = \frac{\mu(\Omega^c)}{\mu(\Omega)}$.

It is a consequence of a the following more natural inequality where we control only the distance to the local mass $\langle u \rangle_{\mu,\Omega}$.

Lemma 3.22. *Let $\Omega \subset \mathbb{R}^d$ be bounded and $\mu \in L^\infty_+(\Omega)$. Then for all $u \in L^p_\mu$,*

$$0 \;\le\; \int_\Omega u^{p-1}(u - \langle u \rangle_{\mu,\Omega})\mu \;\le\; C_{\mathrm{PW}} \int_\Omega \mathfrak{D}_p(u)\mu,$$

where $C_{\mathrm{PW}} = \operatorname{diam}(\Omega)^{d+\alpha}\frac{\|\mu\|_{L^\infty(\Omega)}}{\mu(\Omega)}$.

Proof of Lemma 3.22. We normalize μ to have $\mu \in \mathcal{P}(\Omega)$ (space of probability measures). For all $u \in L^p_\mu(\Omega)$ the following identity hold

$$\begin{aligned} 0 \;\le\; \frac{1}{2}\iint_{\Omega^2} (u_* - u)\left(u_*^{p-1} - u^{p-1}\right)\mu_*\mu\,\mathrm{d}x_*\mathrm{d}x \;&=\; \iint_{\Omega^2} u\left(u^{p-1} - u_*^{p-1}\right)\mu_*\mu\,\mathrm{d}x_*\mathrm{d}x \\ &=\; \iint_{\Omega^2} u^{p-1}(u - u_*)\,\mu_*\mu\,\mathrm{d}x_*\mathrm{d}x \\ &=\; \int_\Omega u^{p-1}\left(u - \langle u \rangle_{\mu,\Omega}\right)\mu. \end{aligned}$$

Hence, using that $|x-y| < 2\operatorname{diam}(\Omega)$, we get

$$\begin{aligned} \int_\Omega u^{p-1}\left(u - \langle u \rangle_{\mu,\Omega}\right)\mu \;&=\; \iint_{\Omega^2} \frac{(u_* - u)\,(u_*^{p-1} - u^{p-1})}{2|x - x_*|^{d+\alpha}}|x - x_*|^{d+\alpha}\mu_*\mu\,\mathrm{d}x_*\mathrm{d}x \\ &\le\; \operatorname{diam}(\Omega)^{d+\alpha}\|\mu\|_{L^\infty(\Omega)}\int_{\mathbb{R}^d} \mathfrak{D}_p(u)\,\mu. \end{aligned}$$

It concludes the proof. □

Proof of Proposition 3.21. Since $\langle u\rangle_\mu = 0$, we have

$$\begin{aligned}\int_\Omega |u|^p\mu &= \int_\Omega u^{p-1}(u-\langle u\rangle_{\mu,\Omega})\mu + \left(\int_\Omega u^{p-1}\mu\right)\frac{1}{\mu(\Omega)}\left(\int_\Omega u\mu\right)\\ &= \int_\Omega u^{p-1}(u-\langle u\rangle_{\mu,\Omega})\mu - \frac{1}{\mu(\Omega)}\left(\int_\Omega u^{p-1}\mu\right)\left(\int_{\Omega^c} u\mu\right),\end{aligned}$$

and, by using Hölder's inequality, the second term can be bounded in the following way

$$\begin{aligned}\frac{1}{\mu(\Omega)}\left|\left(\int_\Omega u^{p-1}\mu\right)\left(\int_{\Omega^c} u\mu\right)\right| &\le \frac{1}{\mu(\Omega)}\,\|u\|^{p-1}_{L^p_\mu(\Omega)}\,\mu(\Omega)^{\frac{1}{p}}\,\|u\|_{L^p_\mu(\Omega^c)}\,\mu(\Omega^c)^{\frac{1}{q}}\\ &\le \varepsilon_\Omega\,\|u\|^{p-1}_{L^p_\mu(\Omega)}\,\|u\|_{L^p_\mu(\Omega^c)}.\end{aligned}$$

We apply Lemma 3.22 to conclude. □

3.6.3 Lyapunov + Poincaré method

The following proposition is nothing but the part of Theorem 3.6 concerning $\beta \in (-\alpha, 0)$, leading to polynomial convergence. It is inspired from [16] where a Local Poincaré together with a Foster-Lyapunov condition are used in the case of the classical Laplacian to prove convergence in spaces of the form $L^2(F^{-1/2}M)$ where M is an exponential or polynomial weight. As this technique strongly uses the formula for gradient of the product of two functions, which is not available for the fractional Laplacian, we work in spaces of the form $L^p((\lambda F^{1-p} + m^p)^{1/p})$ instead, and we use the fact that F has polynomial decay at infinity.

Proposition 3.23. *Assume $\beta \in (-\alpha, 0)$. Let $m = \langle x\rangle^k$ and $\bar m = \langle x\rangle^{\bar k}$ with $|\beta| < k < \bar k < \alpha \wedge 1$ and $f \in L^p(\bar m)$ be a solution to the (FFP) equation for $p \in \left(1, 1+\frac{k-|\beta|}{d+\alpha-k} \wedge p_\beta\right)$ and $p < \frac{\alpha}{k}$ if $k > \alpha - \beta$. Then, the following polynomial convergence holds*

$$\|f - F\|_{L^p(m)} \lesssim \frac{1}{\langle t\rangle^{(\bar k - k)/|\beta|}}\|f^{\rm in} - F\|_{L^p(\bar m)}.$$

When $p = 1$, the same convergence holds replacing with rate $\langle t\rangle^{-a/|\beta|}$ for any $a < \bar k - k$.

Proof of Proposition 3.23. Assume first that $p > 1$. By replacing f by $f - F$ and by conservation of the mass, we can assume $\langle f\rangle_{\mathbb{R}^d} = 0$. By Proposition 3.14, $F \gtrsim \frac{c}{\langle x\rangle^{d+\alpha}}$ and for $p \in (1, p_\alpha)$ with $p'_\alpha = \frac{d+\alpha}{k}$, we have $\varepsilon_0 := kp - (p-1)(d+\alpha) > 0$. Therefore, we have

$$F^{1-p} \lesssim \frac{m^p}{\langle x\rangle^{\varepsilon_0}}, \tag{3.53}$$

and we deduce that $f \in L^p(F^{-1/q})$. Moreover $F \in L^1_+$ and $F \in L^\infty(m)$ from Proposition 3.12. Therefore, if $f \in L^p(F^{-1/q})$, by combining the fractional Poincaré-Wirtinger inequality (Proposition 3.21) with (3.52), we get for a given $\Omega \subset \mathbb{R}^d$ bounded

$$C_{\rm PW}\,\partial_t\left(\int_{\mathbb{R}^d}|f|^pF^{1-p}\right) \le -\int_\Omega |f|^pF^{1-p} + \varepsilon_\Omega\,\|f\|^{p-1}_{L^p_{F^{1-p}}(\Omega)}\,\|f\|_{L^p_{F^{1-p}}(\Omega^c)}. \tag{3.54}$$

Moreover, from estimates (3.27) and (3.30), for $kp < \alpha$, we have

$$\frac{1}{p}\partial_t \left(\int_{\mathbb{R}^d} |f|^p m^p\right) \;\leq\; \int_{\mathbb{R}^d} |f|^p m^p \left(\frac{C}{\langle x\rangle^{\alpha}} + \varphi_{m,p}\right).$$

Or we can also use estimates (3.27) and (3.28) to deduce that for $k < \alpha$,

$$\frac{1}{p}\partial_t \left(\int_{\mathbb{R}^d} |f|^p m^p\right) \;\leq\; \int_{\mathbb{R}^d} |f|^p m^p \left(\frac{C}{\langle x\rangle^{\alpha-k}} + \varphi_{m,p}\right).$$

From (3.9) and one of the two above estimates, if $kp < \alpha$ or $|\beta| < \alpha - k$, we get the Foster-Lyapunov like estimate

$$\frac{1}{p}\partial_t \left(\int_{\mathbb{R}^d} |f|^p m^p\right) \;\leq\; \int_{\mathbb{R}^d} |f|^p \left(b\mathbb{1}_\Omega - a\,\langle x\rangle^{kp+\beta}\right), \tag{3.55}$$

for a given $(a,b) \in \mathbb{R}^2_+$. We define $M^p := m^p + \lambda C_{\mathrm{PW}} F^{1-p}$ in order to use the negative part of both estimates (3.54) and (3.55). Adding the two expressions, we obtain

$$\partial_t \left(\int_{\mathbb{R}^d} |f|^p M^p\right) \;\leq\; \int_{\mathbb{R}^d} |f|^p \left((b - \lambda F^{1-p})\mathbb{1}_\Omega + \lambda\varepsilon_\Omega F^{1-p} - a\,\langle x\rangle^{kp+\beta}\right).$$

Using the fact that $F \in L^\infty(m)$ and (3.53), we obtain the existence of $c > 0$ depending only on F such that

$$\partial_t \left(\int_{\mathbb{R}^d} |f|^p M^p\right) \;\leq\; \int_{\mathbb{R}^d} |f|^p \left((b - \lambda c)\mathbb{1}_\Omega + \langle x\rangle^{kp}\left(\lambda\varepsilon_\Omega c\,\langle x\rangle^{-\varepsilon_0} - a\,\langle x\rangle^{-|\beta|}\right)\right).$$

Now we remark that since $|\beta| < k$, for $p < \frac{d+\alpha-|\beta|}{d+\alpha-k} = 1 + \frac{k-|\beta|}{d+\alpha-k}$, we obtain $-\varepsilon_0 < \beta$. Taking also $\lambda > \frac{b}{c}$, we get

$$\partial_t \left(\int_{\mathbb{R}^d} |f|^p M^p\right) \;\leq\; \int_{\mathbb{R}^d} |f|^p \,\langle x\rangle^{kp+\beta} \left(\lambda\varepsilon_\Omega c - a\right).$$

Thus, by taking Ω large enough so that $\lambda\varepsilon_\Omega c - a < 0$, we obtain the existence of $\bar{a} > 0$ such that

$$\partial_t \left(\int_{\mathbb{R}^d} |f|^p M^p\right) \;\leq\; -\bar{a}\int_{\mathbb{R}^d} |f|^p \,\langle x\rangle^{kp+\beta}. \tag{3.56}$$

Let $\bar{k} \in (k, \alpha)$ and $\varepsilon := p(\bar{k} - k) > 0$. By Hölder's inequality, we have

$$\int_{\mathbb{R}^d} |f|^p \,\langle x\rangle^{kp} \;\leq\; \left(\int_{\mathbb{R}^d} |f|^p \,\langle x\rangle^{kp+\beta}\right)^{\varepsilon/(|\beta|+\varepsilon)} \left(\int_{\mathbb{R}^d} |f|^p \,\langle x\rangle^{\bar{k}p}\right)^{|\beta|/(|\beta|+\varepsilon)}.$$

Combining it with (3.56) and Proposition 3.20 leads to

$$\partial_t \left(\int_{\mathbb{R}^d} |f|^p M^p\right) \;\lesssim\; -\bar{a}\left(\int_{\mathbb{R}^d} |f|^p M^p\right)^{1+|\beta|/\varepsilon} \left(\int_{\mathbb{R}^d} |f^{\mathrm{in}}|^p \,\langle x\rangle^{\bar{k}p}\right)^{-|\beta|/\varepsilon},$$

where we used that by (3.53) and the positivity of F, we have $\langle x\rangle^{kp} \leq M^p \lesssim \langle x\rangle^{kp}$. By Grönwall's inequality, we obtain

$$\int_{\mathbb{R}^d} |f|^p m^p \;\leq\; \int_{\mathbb{R}^d} |f|^p M^p \;\lesssim\; \frac{1}{t^{\varepsilon/|\beta|}} \int_{\mathbb{R}^d} |f^{\mathrm{in}}|^p \,\langle x\rangle^{\bar{k}p}, \tag{3.57}$$

which gives the expected result. When $p = 1$, we use the regularization from L^1 to L^p proved in Proposition 3.9 to get for $t = 1$

$$\|f(t=1)\|_{L^p(\tilde m)} \lesssim \|f^{\rm in}\|_{L^1(\tilde m)},$$

for p sufficiently close to 1 in order to ensure that $L^p(m) \hookrightarrow L^1(\tilde m)$ with $\tilde m = \langle x\rangle^{\tilde k}$ with $\tilde k \in (0,k)$. Then we use (3.57) to obtain for $t > 1$

$$\|f\|_{L^1(\tilde m)} \lesssim \|f\|_{L^p(m)} \lesssim \frac{1}{t^{\varepsilon/p|\beta|}}\|f(t=1)\|_{L^p(\tilde m)} \lesssim \frac{1}{t^{\varepsilon/p|\beta|}}\|f^{\rm in}\|_{L^1(\tilde m)},$$

which ends the proof. □

3.7 Exponential Convergence to the equilibrium for $\beta \geq 0$

When $\beta \geq 0$, the confinement is sufficiently strong to get an exponential time decay toward equilibrium for $|x|$ large. To get the local behavior, instead of using a local Poincaré inequality as in previous section, we will use the gain of positivity from Proposition 3.14.

Proposition 3.24 (Convergence in $L^1(m)$)**.** *Assume $\beta \geq 0$ and let f be a solution of the* (FFP) *equation with $f^{\rm in} \in L^1(m)$. Then, there exists $\bar a > 0$ such that for any $t \in \mathbb{R}_+$*

$$\|f(t) - F\|_{L^1(m)} \;\leq\; e^{-\bar a t}\|f^{\rm in} - F\|_{L^1(m)}.$$

Proof of Proposition 3.24. We want here to use the strategy from Hairer and Mattingly in [112] so that we use the following notations $P_t := e^{t\mathsf{L}^*}$, $X := L^1(m)$ and $X' = L^\infty(m^{-1})$ where $m = \langle x\rangle^k$ with $k \in (0, \alpha \wedge 1)$. We recall that from Theorem 3.1 we immediately deduce by duality that $P : \mathbb{R}_+ \to \mathscr{B}(X')$ is a positive C^0-semigroup such that $P_t 1 = 1$. The strategy consists in proving the following Lyapunov and positivity conditions.

Step 1. Lyapunov condition. Since $E \cdot x \gtrsim |x|^2$, by Proposition 3.2, we have

$$\mathsf{L}^* m \;=\; I(m) - E\cdot\nabla m \;\leq\; b - am.$$

Moreover, by using Duhamel's formula, we have

$$e^{(\mathsf{L}^*+a)t} \;=\; m + e^{(\mathsf{L}^*+a)t} \star (\mathsf{L}^* + a)m.$$

Therefore, we obtain

$$e^{at}P_t m \;\leq\; m + \int_0^t e^{as}P_s b\,{\rm d}s \;\leq\; m + be^{at}/a,$$

from what we deduce

$$P_t m \;\leq\; \gamma_t m + c, \tag{3.58}$$

with $c = b/a$ and $\gamma_t = e^{-at} \in (0,1)$.

Step 2. Positivity condition. From Proposition 3.14, we know that there exists $\nu_t(x) = \nu(t,x) \in L^\infty(\mathbb{R}_+, L^1_+(m))$ strictly positive such that for any $f \in L^1_+(m)$, we have

$$e^{t\mathsf{L}} f \;\geq\; \nu_t \int_{B_R} f.$$

By duality, it implies that

$$P_t \geq \langle \nu_t, \cdot \rangle \mathbb{1}_{m(x)<r}, \tag{3.59}$$

where $r = m(R)$.

Step 3. Convergence in $L^1(m)$. We define $m_\lambda := 1+\lambda m$ and the following seminorm on $L^\infty(m^{-1})$

$$|\varphi|_{\dot{L}^\infty(m_\lambda^{-1})} \;:=\; \sup_{(x,y)\mathbb{R}^{2d}} \left(\frac{|\varphi(x)-\varphi(y)|}{m_\lambda(x)+m_\lambda(y)} \right).$$

Then as proved in [112, Lemma 2.1], we have

$$|\varphi|_{\dot{L}^\infty(m_\lambda^{-1})} \;=\; \inf_{c\in\mathbb{R}} \|\varphi - c\|_{L^\infty(m_\lambda^{-1})}. \tag{3.60}$$

Moreover, [112, Theorem 3.1] tells us that since (3.58) and (3.59) imply that for any fixed time $t > 0$ there exists a constant $\bar{\gamma}_t \in (0,1)$ such that

$$|P_t\varphi|_{\dot{L}^\infty(m_\lambda^{-1})} \;\leq\; \bar{\gamma}_t |\varphi|_{\dot{L}^\infty(m_\lambda^{-1})}. \tag{3.61}$$

By using the semigroup property, we obtain that the optimal $\bar{a}_t := -\ln(\bar{\gamma}_t) > 0$ verifies $\bar{a}_{t+s} \geq \bar{a}_t + \bar{a}_s$, from what we deduce the existence of $\bar{a} > 0$ such that

$$\inf_{c\in\mathbb{R}} \|P_t\varphi - c\|_{X'} \;\leq\; e^{-\bar{a}t}\|\varphi\|_{X'},$$

where we replaced $L^\infty(m_\lambda^{-1})$ by X' by equivalence of the norms. Take a sequence $(c_n)_{n\in\mathbb{N}}$ converging to the minimizer. Then, we can write for $f^{\rm in} \in L^1(m)$ such that $\langle f^{\rm in}\rangle_{\mathbb{R}^d} = 0$

$$\begin{aligned} \langle e^{t\mathsf{L}} f^{\rm in}, \varphi \rangle_{X,X'} &= \langle f^{\rm in}, P_t\varphi - c_n \rangle_{X,X'} \\ &\leq \|f^{\rm in}\|_{L^1(m)} \|P_t\varphi - c_n\|_{X'}. \end{aligned}$$

Passing to the limit $n \to \infty$, for $f(t) := e^{t\mathsf{L}} f^{\rm in}$, we get

$$\|f\|_{L^1(m)} \;=\; \sup_{\|\varphi\|_{X'}\leq 1} \langle f, \varphi \rangle_{X,X'} \;\leq\; e^{-\bar{a}t}\|f^{\rm in}\|_{L^1(m)}.$$

Proposition 3.24 follows by taking $f - F$ instead of f. □

Proof of Theorem 3.6. The part concerning polynomial convergence when $\beta \in (-\alpha, 0)$ was proved in Proposition 3.23. Therefore we just have to prove the part concerning exponential convergence when $\alpha \geq 2$. Thanks to the regularization property of the semigroup from $L^1(m)$ to $L^p(m)$ as proved in Proposition 3.9, we know that

$$\|f - F\|_{L^p(m)} \lesssim \left(c + t^{-\frac{d}{q\alpha}} \right) e^{t\lambda_1} \|f^{\rm in} - F\|_{L^1(m)},$$

where λ_1 is exactly such that

$$\|f - F\|_{L^1(m)} \lesssim e^{t\lambda_1} \|f^{\rm in} - F\|_{L^1(m)}.$$

From Proposition 3.24, we deduce that $\lambda_1 = -\bar{a} < 0$, which gives the result. □

Chapter 4

Hypocoercivity with sub-exponential local equilibria

Abstract

Hypocoercivity methods are applied to linear kinetic equations without any space confinement, when local equilibria have a sub-exponential decay. By Nash type estimates, global rates of decay are obtained, which reflect the behavior of the macroscopic Fokker-Planck equation obtained in the diffusion limit. The method applies to Fokker-Planck and scattering collision operators. The main tools are a weighted Poincaré inequality and norms with various weights. The corresponding estimates are new and not limited to sub-exponential equilibria.

Résumé

On utilise des méthodes hypocoercives sur des équations cinétiques linéaires sans confinement et dont l'équilibre local a une décroissance sous-exponentielle. AU moyen d'estimées de type Nash, on obtient des taux de décroissance globaux qui reflètent le comportement de l'équation de Fokker-Planck que l'on obtient à la limite de diffusion. La méthode s'applique aux opérateurs de Fokker-Planck et de Boltzmann linéaire. Les principaux outils sont des normes et une inégalité de Poincaré à poids. Les estimées correspondantes sont nouvelles et ne sont pas limitées aux équilibres sous-exponentiels.

Contents

4.1 Introduction

This chapter is devoted to the decay rates in weighted L^2 norms of the solution to the Cauchy problem

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mathsf{L} f\,, \\ f(0,x,v) = f^{\mathrm{in}}(x,v)\,, \end{cases} \tag{4.1}$$

for a distribution function $f(t,x,v)$, with *position* $x \in \mathbb{R}^d$, *velocity* $v \in \mathbb{R}^d$, and *time* $t \geq 0$. The collision operator L acts only on the velocity variable and it is assumed to be such that its null space is spanned by the *local equilibrium*

$$\forall\, v \in \mathbb{R}^d, \quad F(v) = C_\gamma\, e^{-\langle v\rangle^\gamma} \quad \text{with} \quad \langle v\rangle := \sqrt{1+|v|^2}.$$

We are interested in the regime with *sub-exponential* local equilibria corresponding to

$$\gamma \in (0,1). \tag{4.2}$$

Local equilibria also appear in the literature as *microscopic equilibria*. We assume that C_γ is a normalization constant such that F is a probability density. More specifically, we have in mind two linear collision operators: the *Fokker-Planck* operator

$$\mathsf{L}_1 f = \nabla_v \cdot \Big(F\, \nabla_v \big(F^{-1} f\big)\Big),$$

and a *linear Boltzmann* or *scattering* collision operator

$$\mathsf{L}_2 f = \int_{\mathbb{R}^d} \mathrm{b}(\cdot, v') \Big(f(v')\, F(\cdot) - f(\cdot)\, F(v')\Big)\, \mathrm{d}v'.$$

Under assumptions on the *cross-section* b that will be given later, these two operators L_1 and L_2 are responsible for the same asymptotic behavior. The operator L_1 is a local operator with sharper estimates, while L_2 is non-local but we have more freedom on the choice of b, with the drawback that estimates are not as tight as for L_1.

We will denote the measure associated with the inverse of the local equilibrium by

$$\mathrm{d}\mu(v) := F(v)^{-1}\,\mathrm{d}v.$$

The easiest case to consider is of course the so-called *homogeneous* case

$$f(t,x,v) = g(t,v),$$

in which the distribution function is independent of x. If we assume that $\mathsf{L} = \mathsf{L}_1$, then (4.1) is reduced to a simple Fokker-Planck equation and by a standard computation, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |g-\overline{g}|^2\,\mathrm{d}\mu = -\,2\int_{\mathbb{R}^d}\left|\nabla_v\Big(F^{-1}\,g\Big)\right|^2\mathrm{d}\mu, \tag{4.3}$$

where $\overline{g} := \big(\int_{\mathbb{R}^d} g\,F\,\mathrm{d}\mu\big)\,F = \big(\int_{\mathbb{R}^d} g\,\mathrm{d}v\big)\,F$. If $\gamma \geq 1$, it is easy to conclude using a Poincaré inequality which asserts that $\int_{\mathbb{R}^d}\left|\nabla_v\Big(F^{-1}\,g\Big)\right|^2\mathrm{d}\mu \;\leq\; \lambda\int_{\mathbb{R}^d}|g-\overline{g}|^2\,\mathrm{d}\mu$ for some positive constant λ and deduce that $\int_{\mathbb{R}^d}|g-\overline{g}|^2\,\mathrm{d}\mu$ is decaying at an exponential rate. However, we are interested in the range (4.2) and there is a different type of Poincaré inequalities, namely

$$\forall\,g\in\mathcal{D}(\mathbb{R}^d)\,,\quad \int_{\mathbb{R}^d}\left|\nabla_v\Big(F^{-1}\,g\Big)\right|^2\mathrm{d}\mu \geq \mathcal{C}\int_{\mathbb{R}^d}|g-\overline{g}|^2\,\langle v\rangle^{2\,(\gamma-1)}\,\mathrm{d}\mu. \tag{4.4}$$

The point is that there is now a weight in the right-hand side: see Appendix B.1 for details. Under appropriate conditions on the initial data, this is enough to obtain that $\int_{\mathbb{R}^d}|g-\overline{g}|^2\,\mathrm{d}\mu$ has an algebraic decay rate: a short proof can be found in Appendix B.2 and we refer to [130] for detailed results in various L^p norms. Estimates based on *weak Poincaré inequalities* are also very popular in the semi-group and Markov processes scientific community: see [184] and [17, Proposition 7.5.10], but with the disadvantage that a uniform estimate of the solution has to be assumed.

Our purpose is to consider the *non-homogeneous* case of (4.1) with a non-negative initial datum $f^{\rm in}\in L^1(\mathbb{R}^d\times\mathbb{R}^d)$. We adapt the L^2 hypocoercivity method of [80, 81]. This method was initially considered in presence of a potential of confinement guaranteeing a macroscopic Poincaré inequality and later extended to the case of a weaker inequality as in [57]. In the case of (4.1), the mass is conserved and there is no stationary state with finite mass, so we already know that $f(t,\cdot,\cdot)$ locally vanishes. We aim at estimating the rate of decay of $\iint_{\mathbb{R}^{2d}}|f(t,x,v)|^2\,\mathrm{d}x\,\mathrm{d}\mu$ as $t\to+\infty$. In the range $\gamma\geq 1$, an estimate has been obtained in [44], which relies on Nash's inequality. In the range (4.2), we shall prove in this chapter that, at microscopic level, the Poincaré inequality can be replaced by (4.4) in case of L_1 and by a similar inequality in case of L_2, while at macroscopic level, we still rely on Nash's inequality. To state a result, we need some notations and a few additional assumptions in the case of the scattering operator. Let us consider the norms

$$\|f\|_k = \left(\iint_{\mathbb{R}^{2d}}|f|^p\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\right)^{1/2}\quad\text{and}\quad \|f\| = \|f\|_0\,.$$

We also define on $L^2(\mathrm{d}x\,\mathrm{d}\mu)$ the scalar product such that $\|f\|^2 = \langle f, f\rangle$ by

$$\langle f_1, f_2\rangle := \iint_{\mathbb{R}^{2d}} f_1\, f_2\, \mathrm{d}x\, \mathrm{d}\mu.$$

With these notations in hand, we can list our hypotheses. We first assume *local mass conservation*

$$\int_{\mathbb{R}^d} \mathsf{L} f\, \mathrm{d}v = 0.$$

Such a property is always granted if $\mathsf{L} = \mathsf{L}_1$ and holds when $\mathsf{L} = \mathsf{L}_2$ if b satisfies

$$\int_{\mathbb{R}^d} \Big(\mathrm{b}(v, v') - \mathrm{b}(v', v)\Big)\, F'\, \mathrm{d}v' = 0, \tag{H1}$$

where F' stands for $F(v')$. Notice that *micro-reversibility*, *i.e.*, the symmetry of b, is not required. We assume next that b takes nonnegative values and is such that the *collision frequency* ν verifies

$$\nu(v) := \int_{\mathbb{R}^d} \mathrm{b}(v, v')\, F'\, \mathrm{d}v' = G(v)\, \langle v\rangle^{\beta} \tag{H2}$$

for some $\beta \le 0$ and for some positive, continuous function G such that $\lim_{|v|\to\infty} G(v) = 1$. In case $\mathsf{L} = \mathsf{L}_2$, Inequality (4.4) has to be replaced by an analogous inequality. We shall assume that there exists a finite positive constant $\mathcal{C}$ such that

$$\forall\, h \in L^2(\mathbb{R}^d, \mathrm{d}\mu)\,, \quad \int_{\mathbb{R}^d} \Big|h - \widetilde{h}\Big|^2 \nu\, F\, \mathrm{d}v \le \mathcal{C} \iint_{\mathbb{R}^{2d}} \mathrm{b}(v, v')\, \Big|h' - h\Big|^2\, F\, F'\, \mathrm{d}v\, \mathrm{d}v' \tag{4.5}$$

where $\widetilde{h} = \int_{\mathbb{R}^d} h\, \mathrm{d}\mu$. Sufficient conditions on b can be found for instance in [73, Proposition 2.2] or in [161, Lemma 1]. Notice that a Cauchy-Schwarz inequality yields

$$\begin{aligned}\int_{\mathbb{R}^d} \Big|h - \widetilde{h}\Big|^2 \nu\, F\, \mathrm{d}v &= \int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} \Big(h - h'\Big)\, F'\, \mathrm{d}v'\right|^2 \nu\, F\, \mathrm{d}v \\ &\le \int_{\mathbb{R}^d} \frac{F}{\nu}\, \mathrm{d}v \iint_{\mathbb{R}^{2d}} \Big|h - h'\Big|^2 \nu\, \nu'\, F\, \mathrm{d}v\, F'\, \mathrm{d}v'\end{aligned}$$

so that a sufficient condition is obtained by assuming

$$\sup_{(v,v')\in\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{\nu(v)\, \nu(v')}{\mathrm{b}(v, v')}\right) < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{F(v)}{\nu(v)}\, \mathrm{d}v < \infty \tag{H3}$$

which does not require the local integrability of b. Other sufficient conditions can be found in [161]. We shall additionally assume that b satisfies

$$\left\|\int_{\mathbb{R}^d} \mathrm{b}(v, v')\, \Big(v' - v\Big)\, F\, F'\, \mathrm{d}v'\right\|_{L^2\left(\langle v\rangle^{-\beta} \mathrm{d}\mu\right)} < \infty \quad \text{and} \quad \mathcal{C}_{\mathrm{b}} := \sup_{v\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathrm{b}(v', v)^2}{\nu(v)^2}\, F'\, \mathrm{d}v' < \infty. \tag{H4}$$

Our main result is concerned with the rate of decay to equilibrium of the solutions of (4.1).

Theorem 4.1. *Let γ be such that (4.2) holds and take $k > 0$. Assume that either $\mathsf{L} = \mathsf{L}_1$ and $\beta = 2(\gamma - 1)$, or $\mathsf{L} = \mathsf{L}_2$ and (H1)–(H4) hold. There exists a constant $C > 0$ such that any solution f of (4.1) with initial datum $f^{\rm in} \in L^2\left(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\right) \cap L^1(\mathrm{d}x\,\mathrm{d}v)$ satisfies*

$$\forall\, t \geq 0\,, \quad \|f(t,\cdot,\cdot)\|^2 = \iint_{\mathbb{R}^{2d}} \left|f(t,x,v)\right|^2 \mathrm{d}x\,\mathrm{d}\mu \leq C\,\frac{\|f^{\rm in}\|_k^2 + \|f^{\rm in}\|_{L^1(\mathrm{d}x\,\mathrm{d}v)}^2}{(1+t)^a}$$

with rate $a = \min\{d/2, k/|\beta|\}$.

If $k \geq d|\beta|/2$, we recover the rate of decay of the heat equation as in the case $\gamma \geq 1$. This makes sense because our L^2 hypocoercivity method is designed to capture the rate of the diffusion limit as in [44, 81]. However, if $k \in (0, d|\beta|/2)$, a new limitation appears due to the lower rate of relaxation towards the local equilibrium: see Appendix B.2 for details.

This chapter is organized as follows. In Section 4.2, we prove an estimate which relates an entropy which is equivalent to $\|f\|^2$ to an entropy production term involving a microscopic and a macroscopic component. Using weighted L^2 estimates established in Section 4.3, we obtain a new control by the microscopic component in Lemma 4.7 while the macroscopic component is estimated as in [44] using Nash's inequality, see Lemma 4.6. By collecting these estimates in Section 4.4, we establish the proof of Theorem 4.1. Two appendices are devoted to $\mathsf{L} = \mathsf{L}_1$: in Appendix B.1 we provide a new proof of (4.4) and comment on the interplay with weak Poincaré inequalities, while the homogeneous case is dealt with in Appendix B.2 and rates of relaxation towards the local equilibrium are discussed on the basis of weighted L^2 norms. The main novelty of our approach is that we use new interpolations in order to exploit the entropy production term and that, with the appropriate weights, no other norm is needed than weighted L^2 norms.

4.2 An entropy–entropy production estimate

We adapt the strategy of [81, 44]. Let Π be the orthogonal projection operator on $\mathrm{Ker}(\mathsf{L})$ in $L^2(\mathbb{R}^d, \mathrm{d}\mu)$, defined by

$$\Pi f := \rho_f\, F \quad \text{where} \quad \rho_f := \int_{\mathbb{R}^d} f\, \mathrm{d}v.$$

To build a suitable Lyapunov functional we introduce the operator

$$\mathsf{A} := \left(1 + |\mathsf{T}\Pi|^2\right)^{-1} (\mathsf{T}\Pi)^*,$$

where we used the notation $|\mathsf{B}|^2 = \mathsf{B}^*\mathsf{B}$, and consider

$$\mathsf{H}[f] := \frac{1}{2}\,\|f\|^2 + \delta\,\langle \mathsf{A}f, f\rangle\,.$$

It is known from [81, Lemma 1] that for $\delta \in (0,1)$, $\mathsf{H}[f]$ and $\|f\|^2$ determine equivalent norms, in the sense that

$$\frac{1}{2}\,(1-\delta)\,\|f\|^2 \leq \mathsf{H}[f] \leq \frac{1}{2}\,(1+\delta)\,\|f\|^2. \tag{4.6}$$

A direct computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}[f] = -\,\mathsf{D}[f] \tag{4.7}$$

$$\text{with}\quad \mathsf{D}[f] :=-, \langle \mathsf{L}f, f\rangle + \delta\,\langle \mathsf{AT\Pi}f, \mathsf{\Pi}f\rangle \\ + \delta\,\langle \mathsf{AT}(1-\mathsf{\Pi})f, \mathsf{\Pi}f\rangle - \,\delta\,\langle \mathsf{TA}(1-\mathsf{\Pi})f, (1-\mathsf{\Pi})f\rangle - \delta\,\langle \mathsf{AL}(1-\mathsf{\Pi})f, \mathsf{\Pi}f\rangle$$

where we have used that $\langle \mathsf{A}f, \mathsf{L}f\rangle = 0$. As a consequence of (4.4) and (4.5), there is a positive constant $\mathcal{C}$ such that

$$\langle \mathsf{L}f, f\rangle \le -\,\mathcal{C}\,\|(1-\mathsf{\Pi})f\|_\beta^2\,. \tag{4.8}$$

Proposition 4.2. *Under the assumptions of Theorem 4.1, there exists* $\kappa \in (0,1)$ *such that, for any* $f \in L^2\left(\langle v\rangle^\beta\,\mathrm{d}x\,\mathrm{d}\mu\right) \cap L^1(\mathrm{d}x\,\mathrm{d}v)$,

$$\mathsf{D}[f] \ge \kappa\left(\|(1-\mathsf{\Pi})f\|_\beta^2 + \langle \mathsf{AT\Pi}f, \mathsf{\Pi}f\rangle\right).$$

Notice that κ does not depend on $k > 0$. Expressing $\mathsf{D}[f]$ in terms of $\langle \mathsf{AT\Pi}f, \mathsf{\Pi}f\rangle$ has already been used in [44] but the use of the weighted norm $\|(1-\mathsf{\Pi})f\|_\beta$ is a new idea.

Proof. We have to prove that the three last terms in $\mathsf{D}[f]$ are controlled by the first two. The main difference with [81, 44] is the additional weight $\langle v\rangle^\beta$ in the velocity variable.

Step 1: rewriting $\langle \mathsf{AT\Pi}f, \mathsf{\Pi}f\rangle$**.** Let $u = u_f$ be such that $u\,F = (1+|\mathsf{T\Pi}|^2)^{-1}\,\mathsf{\Pi}f$. Then u solves $(u - \Theta\,\Delta u)\,F = \mathsf{\Pi}f$, that is,

$$u - \Theta\,\Delta u = \rho_f, \tag{4.9}$$

where $\Theta := \int_{\mathbb{R}^d} |v\cdot\mathsf{e}|^2\,F(v)\,\mathrm{d}v$ for an arbitrary unit vector e. Since

$$\begin{aligned}\mathsf{AT\Pi}f &= \left(1+|\mathsf{T\Pi}|^2\right)^{-1}(\mathsf{T\Pi})^*(\mathsf{T\Pi})\,\mathsf{\Pi}f\\ &= \left(1+|\mathsf{T\Pi}|^2\right)^{-1}\left(1+|\mathsf{T\Pi}|^2-1\right)\mathsf{\Pi}f\\ &= \mathsf{\Pi}f - \left(1+|\mathsf{T\Pi}|^2\right)^{-1}\mathsf{\Pi}f = \mathsf{\Pi}f - u\,F = (\rho_f - u)\,F\,,\end{aligned}$$

then by using equation (4.9), we obtain

$$\langle \mathsf{AT\Pi}f, \mathsf{\Pi}f\rangle = \langle \mathsf{\Pi}f - u\,F, \mathsf{\Pi}f\rangle = \langle -\Theta\,\Delta u\,F, (u - \Theta\,\Delta u)\,F\rangle\,,$$

from which we deduce

$$\langle \mathsf{AT\Pi}f, \mathsf{\Pi}f\rangle = \Theta\,\|\nabla u\|_{L^2(\mathrm{d}x)}^2 + \Theta^2\,\|\Delta u\|_{L^2(\mathrm{d}x)}^2\,. \tag{4.10}$$

Step 2: a bound on $\langle \mathsf{AT}(1-\Pi)f, \Pi f\rangle$***.*** We use the fact that

$$\mathsf{A}^*\Pi f = \mathsf{T}\Pi u\, F = \mathsf{T} u\, F, \tag{4.11}$$

to compute

$$\langle \mathsf{AT}(1-\Pi)f, \Pi f\rangle = \langle (1-\Pi)f, \mathsf{T}^*\mathsf{A}^*\Pi f\rangle = \langle (1-\Pi)f, \mathsf{T}^*\mathsf{T} u\, F\rangle\,.$$

Therefore, since $\mathsf{T}^*\mathsf{T}u\,F = -v\cdot\nabla_x\,(v\cdot\nabla_x u)\,F$, the Cauchy-Schwarz inequality yields

$$\begin{aligned} |\langle \mathsf{AT}(1-\Pi)f, \Pi f\rangle| &\le \|(1-\Pi)f\|_\beta \left\| \sqrt{F}\,\langle v\rangle^{-\frac{\beta}{2}}\, v\cdot\nabla_x\,(v\cdot\nabla_x u)\,\sqrt{F}\right\|_{L^2(\mathrm{d}x\,\mathrm{d}v)} \\ &\le \Theta_{4-\beta}\,\|(1-\Pi)f\|_\beta\,\|\Delta u\|_{L^2(\mathrm{d}x)} \\ &\le \mathcal{C}_4\,\|(1-\Pi)f\|_\beta\,\langle \mathsf{AT}\Pi f, \Pi f\rangle^{\frac12}\,. \end{aligned} \tag{4.12}$$

where we have used Identity (4.10), $\mathcal{C}_4 = \Theta_{4-\beta}/\Theta$ and $\Theta_k := \int_{\mathbb{R}^d}\langle v\rangle^k\,F(v)\,\mathrm{d}v$. With this convention, notice that $\Theta_2 = d\,\Theta$.

Step 3: estimating $\langle \mathsf{TA}(1-\Pi)f, (1-\Pi)f\rangle$***.*** As noticed in [81, Lemma 1], the equation $g = \Pi g = \mathsf{A}f$ is equivalent to

$$\Big(1 + |\mathsf{T}\Pi|^2\Big) g = (\mathsf{T}\Pi)^* f,$$

which, after multiplying by g and integrating, yields

$$\begin{aligned} \|g\|^2 + \|\mathsf{T}g\|^2 = \Big\langle g, g + |\mathsf{T}\Pi|^2 g\Big\rangle &= \langle g, (\mathsf{T}\Pi)^* f\rangle \\ &= \langle \mathsf{T}\Pi g, f\rangle = \langle \mathsf{TA}f, f\rangle \le \|(1-\Pi)f\|_\beta\,\|\mathsf{TA}f\|_{-\beta} \end{aligned}$$

by the Cauchy-Schwarz inequality. We know that $(\mathsf{T}\Pi)^* = -\,\Pi\mathsf{T}$ so that $\mathsf{A}f = g = w\,F$ is determined by the equation

$$w - \Theta\,\Delta w = -\,\nabla_x\cdot\int_{\mathbb{R}^d} v\,f\,\mathrm{d}v.$$

After multiplying by w and integrating in x, we obtain that

$$\Theta\int_{\mathbb{R}^d}|\nabla_x w|^2\,\mathrm{d}x \le \int_{\mathbb{R}^d}|w|^2\,\mathrm{d}x + \Theta\int_{\mathbb{R}^d}|\nabla_x w|^2\,\mathrm{d}x \le \left(\int_{\mathbb{R}^d}|\nabla_x w|^2\,\mathrm{d}x\right)^{\frac12}\left(\int_{\mathbb{R}^d}\left|\textstyle\int_{\mathbb{R}^d} v\,f\,\mathrm{d}v\right|^2\,\mathrm{d}x\right)^{\frac12},$$

and notice that

$$\int_{\mathbb{R}^d}\left|\textstyle\int_{\mathbb{R}^d} v\,f\,\mathrm{d}v\right|^2\,\mathrm{d}x = \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\langle v\rangle^{\frac{\beta}{2}}\,\frac{(1-\Pi)f}{\sqrt{F}}\cdot|v|\,\langle v\rangle^{-\frac{\beta}{2}}\,\sqrt{F}\,\mathrm{d}v\right|^2\,\mathrm{d}x \le \Theta_{2-\beta}\,\|(1-\Pi)f\|_\beta^2$$

by the Cauchy-Schwarz inequality. Hence

$$\int_{\mathbb{R}^d}|\nabla_x w|^2\,\mathrm{d}x \le \frac{\Theta_{2-\beta}}{\Theta^2}\,\|(1-\Pi)f\|_\beta^2\,,$$

and

$$\|\mathsf{TA}f\|_{-\beta}^2 = \left\|\nabla_x w \cdot (v\ \langle v\rangle^{-\beta/2}\ F)\right\|^2 = \Theta_{2-\beta}\int_{\mathbb{R}^d} |\nabla_x w|^2\,\mathrm{d}x \le \mathcal{C}_2^2\ \|(1-\Pi)f\|_\beta^2\,,$$

with $\mathcal{C}_2 := \Theta_{2-\beta}/\Theta$. Since $g = \mathsf{A}f$ so that $\|\mathsf{A}f\|^2 + \|\mathsf{TA}f\|^2 = \|g\|^2 + \|\mathsf{T}g\|^2$, we obtain that

$$\langle \mathsf{TA}(1-\Pi)f, (1-\Pi)f\rangle = \langle \mathsf{TA}f, f\rangle \le \|(1-\Pi)f\|_\beta\ \|\mathsf{TA}f\|_{-\beta} \le \mathcal{C}_2\ \|(1-\Pi)f\|_\beta^2\,. \quad (4.13)$$

We can also notice that

$$\langle \mathsf{TA}f, f\rangle = \langle (v\cdot\nabla_x w)\,F, f\rangle = \int_{\mathbb{R}^d} \nabla_x w\cdot\Big(\int_{\mathbb{R}^d} v\,f\,\mathrm{d}v\Big)\ \mathrm{d}x = \int_{\mathbb{R}^d} |w|^2\,\mathrm{d}x + \Theta\int_{\mathbb{R}^d} |\nabla_x w|^2\,\mathrm{d}x \ge 0.$$

Step 4: bound for $\langle \mathsf{AL}(1-\Pi)f, \Pi f\rangle$. We use again identity (4.11) to compute

$$\begin{aligned}|\langle \mathsf{AL}(1-\Pi)f, \Pi f\rangle| = |\langle (1-\Pi)f, \mathsf{L}^*\mathsf{A}^*\Pi f\rangle| &= |\langle (1-\Pi)f, \mathsf{L}^*\mathsf{T}u\,F\rangle| \\ &\le \|(1-\Pi)f\|_\beta\ \|\mathsf{L}^*\mathsf{T}u\,F\|_{-\beta}\,.\end{aligned}$$

In case $\mathsf{L} = \mathsf{L}_1$ we remark that

$$\begin{aligned}\|\mathsf{L}^*\mathsf{T}u\,F\|_{-\beta}^2 = \iint_{\mathbb{R}^{2d}} \left|\nabla_v\cdot\Big(F\,\nabla_v\,(v\cdot\nabla_x u)\Big)\right|^2\ \langle v\rangle^{-\beta}\,\mathrm{d}x\,\mathrm{d}\mu &= \iint_{\mathbb{R}^{2d}} |\nabla_v F\cdot\nabla_x u|^2\,\langle v\rangle^{-\beta}\,\mathrm{d}x\,\mathrm{d}\mu \\ &\le \|\nabla_v F\|_{L^2(\langle v\rangle^{-\beta}\mathrm{d}\mu)}^2\ \|\nabla_x u\|_{L^2(\mathrm{d}x)}^2\,.\end{aligned}$$

In case $\mathsf{L} = \mathsf{L}_2$, notice first that

$$\mathsf{LT}u\,F = \Big(\int_{\mathbb{R}^d} \mathrm{b}(\cdot, v')\,(v'-v)\,F'\,\mathrm{d}v'\Big)\,\nabla_x u\,F\,,$$

and thus, by the Cauchy-Schwarz inequality,

$$\|\mathsf{L}^*\mathsf{T}u\,F\|_{-\beta} \le \mathcal{B}\ \|\nabla_x u\|_{L^2(\mathrm{d}x)}\,, \quad \text{where} \quad \mathcal{B} = \left\|\int_{\mathbb{R}^d} \mathrm{b}(v, v')\,(v'-v)\,F'\,F\,\mathrm{d}v'\right\|_{L^2(\langle v\rangle^{-\beta}\mathrm{d}\mu)}$$

is bounded by Assumption (H4). Combining these estimates with identity (4.10) we get

$$|\langle \mathsf{AL}(1-\Pi)f, \Pi f\rangle| \le \mathcal{C}_F\ \|(1-\Pi)f\|_\beta\,\langle \mathsf{AT}\Pi f, \Pi f\rangle^{\frac{1}{2}}\,, \quad (4.14)$$

where $\mathcal{C}_F = \mathcal{B}/\sqrt{\Theta}$.

Step 5: collecting all estimates. Altogether, combining inequalities (4.8) and (4.12)–(4.14), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}[f] \le -\mathcal{C}\ \|(1-\Pi)f\|_\beta^2 &- \delta\,\langle \mathsf{AT}\Pi f, \Pi f\rangle \\ &+ \delta\,(\mathcal{C}_4 + \mathcal{C}_F)\,\|(1-\Pi)f\|_\beta\,\langle \mathsf{AT}\Pi f, \Pi f\rangle^{\frac{1}{2}} + \delta\,\mathcal{C}_2\ \|(1-\Pi)f\|_\beta^2\,,\end{aligned}$$

which by Young's inequality yields the existence of $\kappa > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}[f] \leq -\,\kappa\left(\|(1-\Pi)f\|_\beta^2 + \langle \mathsf{AT}\Pi f, \Pi f\rangle\right)$$

for some $\delta \in (0,1)$. Indeed, with $X := \|(1-\Pi)f\|_\beta$ and $Y := \langle \mathsf{AT}\Pi f, \Pi f\rangle^{\frac{1}{2}}$, it is enough to check that the quadratic form

$$\mathcal{Q}(X,Y) := (\mathcal{C} - \delta\,\mathcal{C}_2)\,X^2 - (\mathcal{C}_4 + \mathcal{C}_F)\,X\,Y + \delta\,Y^2$$

is negative, *i.e.*, $\mathcal{Q}(X,Y) \geq \kappa\,(X^2+Y^2)$ for some $\kappa = \kappa(\delta)$ and $\delta \in (0,1)$. Details are left to the reader. □

4.3 Weighted L^2 estimates

In this section, we show the propagation of weighted norms with power law of arbitrary positive order $k \in \mathbb{R}^+$.

Proposition 4.3. *Let $k > 0$ and f be solution of (4.1) with $f^{\mathrm{in}} \in L^2\Big(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\Big)$. Then there exists a constant $\mathcal{K}_k > 0$ such that*

$$\forall\, t \geq 0 \qquad \|f(t,\cdot,\cdot)\|_{L^2\left(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\right)} \leq \mathcal{K}_k\,\left\|f^{\mathrm{in}}\right\|_{L^2\left(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\right)}.$$

4.3.1 A technical lemma

Lemma 4.4. *If either $\mathsf{L} = \mathsf{L}_1$ or $\mathsf{L} = \mathsf{L}_2$, then*

$$\forall\, t \geq 0\,, \quad \left\|e^{t(\mathsf{L}-\mathsf{T})}\right\|_{L^2(\mathrm{d}x\,\mathrm{d}\mu)\to L^2(\mathrm{d}x\,\mathrm{d}\mu)} \leq 1, \tag{4.15}$$

and there is some $\ell \in \mathbb{R}$ for which, for any $k \geq 0$, there exists $(a_k, b_k, R_k) \in \mathbb{R}\times\mathbb{R}_+^2$ such that

$$\iint_{\mathbb{R}^{2d}} f\,\mathsf{L}f\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu \leq \iint_{\mathbb{R}^{2d}} \left(a_k\,\mathbb{1}_{B_{R_k}} - b_k\,\langle v\rangle^{-\ell}\right)\,|f|^2\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu \tag{4.16}$$

for any $f \in L^2\left(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\right) \cap L^1(\mathrm{d}x\,\mathrm{d}v)$.

Proof. In the Fokker-Planck case $\mathsf{L} = \mathsf{L}_1$, the function $h := f F^{-1}$ solves

$$\partial_t h + v\cdot\nabla_x h = F^{-1}\,\nabla_v\cdot\Big(F\,\nabla_v h\Big)$$

so that

$$\begin{aligned}\iint_{\mathbb{R}^{2d}} f\,\mathsf{L}f\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu &= \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^{2d}} |h(t,v)|^2\,\langle v\rangle^k\,F\,\mathrm{d}x\,\mathrm{d}v\\ &= -\,2\iint_{\mathbb{R}^{2d}} |\nabla_v h|^2\,\langle v\rangle^k\,F\,\mathrm{d}x\,\mathrm{d}v - \iint_{\mathbb{R}^{2d}} \nabla_v(h^2)\cdot\nabla_v\Big(\langle v\rangle^k\Big)\,F\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

This proves (4.15) if $k=0$. Otherwise, (4.16) follows with $\ell = 2-\gamma$ from

$$\nabla_v \log\Big(F\,\nabla_v\big(\langle v\rangle^k\big)\Big) = \frac{k}{\langle v\rangle^4}\Big(d + (k+d-2)\,|v|^2 - \gamma\,\langle v\rangle^\gamma\,|v|^2\Big)\,.$$

In the case of the scattering operator $\mathsf{L} = \mathsf{L}_2$, with $h := f/F$, we have

$$\begin{aligned}
2\int_{\mathbb{R}^d} f\,\mathsf{L}f\,\langle v\rangle^k\,\mathrm{d}\mu &= 2\iint_{\mathbb{R}^{2d}} \mathrm{b}(v,v')\,(h'-h)\,h\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\
&= \iint_{\mathbb{R}^{2d}} \mathrm{b}(v,v')\,((h'-h)\,h + h'h)\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\
&\qquad - \iint_{\mathbb{R}^{2d}} \mathrm{b}(v,v')\,|h|^2\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\,,
\end{aligned}$$

$$\begin{aligned}
2\int_{\mathbb{R}^d} f\,\mathsf{L}f\,\langle v\rangle^k\,\mathrm{d}\mu &= \iint_{\mathbb{R}^{2d}} \mathrm{b}(v,v')\,((h'-h)\,h + h'\,(h-h'))\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\
&\qquad + \iint_{\mathbb{R}^{2d}} \mathrm{b}(v,v')\,\Big(|h'|^2 - |h|^2\Big)\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\
&= -\iint_{\mathbb{R}^{2d}} \mathrm{b}(v,v')|h'-h|^2\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\
&\qquad + \iint_{\mathbb{R}^{2d}} \mathrm{b}(v,v')\,\Big(|h'|^2 - |h|^2\Big)\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\
&\le \iint_{\mathbb{R}^{2d}} \mathrm{b}(v,v')\,\Big(|h'|^2 - |h|^2\Big)\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'.
\end{aligned}$$

However, using Assumption (H1), one can rearrange the last integral

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}} \mathrm{b}(v,v')\,\Big(|h'|^2 - |h|^2\Big)\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\
&= \iint_{\mathbb{R}^{2d}} \mathrm{b}(v,v')\,|h'|^2\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v' - \iint_{\mathbb{R}^{2d}} \mathrm{b}(v',v)\,|h|^2\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\
&= \iint_{\mathbb{R}^{2d}} \mathrm{b}(v,v')\,|h'|^2\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v' - \iint_{\mathbb{R}^{2d}} \mathrm{b}(v,v')\,|h'|^2\,\langle v'\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\
&= \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} \mathrm{b}(v',v)\,\Big(\langle v'\rangle^k - \langle v\rangle^k\Big)\,F'\,\mathrm{d}v'\right]\,|h|^2\,F\,\mathrm{d}v\\
&= \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} \mathrm{b}(v',v)\,\left(\tfrac{\langle v'\rangle^k}{\langle v\rangle^k} - 1\right)\,F'\,\mathrm{d}v'\right]\,|f|^2\,\langle v\rangle^k\,\mathrm{d}\mu.
\end{aligned}$$

This implies Identity (4.15) by taking $k=0$.To get inequality (4.16), we point out that

$$\begin{aligned}
2\int_{\mathbb{R}^d} f\,\mathsf{L}f\,\langle v\rangle^k\,\mathrm{d}\mu &= \int_{\mathbb{R}^d} \mathrm{b}(v',v)\,\tfrac{\langle v'\rangle^k}{\langle v\rangle^k}\,F'\,\mathrm{d}v' - \nu(v)\\
&\le \langle v\rangle^{-k}\,\sqrt{\Theta_{2k}\textstyle\int_{\mathbb{R}^d} \mathrm{b}(v',v)^2\,F'\,\mathrm{d}v'} - \nu(v) \le \Big(\langle v\rangle^{-k}\,\sqrt{\mathcal{C}_\mathrm{b}\,\Theta_{2k}} - 1\Big)\,\nu(v),
\end{aligned}$$

where we used Assumption (H4). Finally, we conclude that inequality (4.16) holds for any $k>0$ with $\ell = -\beta$. □

4.3.2 A splitting method

As in [111, 130, 167], we write $\mathsf{L}-\mathsf{T}$ as a dissipative part C and a bounded part B such that $\mathsf{L}-\mathsf{T}=\mathsf{B}+\mathsf{C}$.

Lemma 4.5. *With the notation of Lemma 4.4, let* $k_1>0$, $k_2>k_1+2\,\ell$, $a=\max(a_{k_1},a_{k_2})$, $R=\max\left\{R_{k_1},R_{k_2}\right\}$, $\mathsf{C}=a\,\mathbb{1}_{B_R}$ *and* $\mathsf{B}=\mathsf{L}-\mathsf{T}-\mathsf{C}$. *For any* $t\in\mathbb{R}_+$, *we have:*

(i) $\|\mathsf{C}\|_{L^2(\mathrm{d}x\,\mathrm{d}\mu)\to L^2\left(\langle v\rangle^{k_2}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\le a\,\langle R\rangle^{k_2/2}$,

(ii) $\|e^{t\mathsf{B}}\|_{L^2\left(\langle v\rangle^{k_1}\,\mathrm{d}x\,\mathrm{d}\mu\right)\to L^2\left(\langle v\rangle^{k_1}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\le 1$,

(iii) $\|e^{t\mathsf{B}}\|_{L^2\left(\langle v\rangle^{k_2}\,\mathrm{d}x\,\mathrm{d}\mu\right)\to L^2\left(\langle v\rangle^{k_1}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\le C\,(1+t)^{-\frac{k_2-k_1}{2\,\ell}}$ *for some* $C>0$.

Proof. Property (i) is a consequence of the definition of C. Property (ii) follows from Lemma 4.4 according to

$$\iint_{\mathbb{R}^{2d}} f\,\mathsf{B}f\;\langle v\rangle^{k_1}\;\mathrm{d}x\,\mathrm{d}\mu\le\iint_{\mathbb{R}^{2d}}\left(a_{k_1}\,\mathbb{1}_{B_{R_{k_1}}}-a\,\mathbb{1}_{B_R}-b_{k_1}\,\langle v\rangle^{-\ell}\right)|f|^2\;\langle v\rangle^{k_1}\;\mathrm{d}x\,\mathrm{d}\mu$$
$$\le-\,b_{k_1}\iint_{\mathbb{R}^{2d}}|f|^2\;\langle v\rangle^{k_1-\ell}\;\mathrm{d}x\,\mathrm{d}\mu.$$

Similarly, we know that $\|e^{t\mathsf{B}}\|_{L^2\left(\langle v\rangle^{k_2}\,\mathrm{d}x\,\mathrm{d}\mu\right)\to L^2\left(\langle v\rangle^{k_2}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\le 1$.

By combining Hölder's inequality

$$\|f\|^2_{L^2\left(\langle v\rangle^{k_1}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\le\|f\|^{\frac{2\,(k_2-k_1)}{k_2-k_1+\ell}}_{L^2\left(\langle v\rangle^{k_1-\ell}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\,\|f\|^{\frac{2\,\ell}{k_2-k_1+\ell}}_{L^2\left(\langle v\rangle^{k_2}\,\mathrm{d}x\,\mathrm{d}\mu\right)}$$

with Property (ii), we obtain

$$\iint_{\mathbb{R}^{2d}} f\,\mathsf{B}f\;\langle v\rangle^{k_1}\;\mathrm{d}x\,\mathrm{d}\mu\le-\,b_{k_1}\,\|f\|^{2\left(1+\frac{\ell}{k_2-k_1}\right)}_{L^2(\langle v\rangle^{k_1}\,\mathrm{d}x\,\mathrm{d}\mu)}\,\left\|f^{\mathrm{in}}\right\|^{-\frac{2\,\ell}{k_2-k_1}}_{L^2(\langle v\rangle^{k_2}\,\mathrm{d}x\,\mathrm{d}\mu)}.$$

With $f=e^{t\mathsf{B}}\,f^{\mathrm{in}}$, Property (iii) follows from Grönwall's lemma according to

$$\|f\|^2_{L^2\left(\langle v\rangle^{k_1}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\le\left(\left\|f^{\mathrm{in}}\right\|^{-\frac{2\,\ell}{k_2-k_1}}_{L^2\left(\langle v\rangle^{k_1}\,\mathrm{d}x\,\mathrm{d}\mu\right)}+\frac{2\,\ell\,b_{k_1}\,t}{k_2-k_1}\left\|f^{\mathrm{in}}\right\|^{-\frac{2\,\ell}{k_2-k_1}}_{L^2\left(\langle v\rangle^{k_2}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\right)^{-\frac{k_2-k_1}{\ell}}$$
$$\le\left(\frac{k_2-k_1}{k_2-k_1+2\,\ell\,b_{k_1}\,t}\right)^{\frac{k_2-k_1}{\ell}}\left\|f^{\mathrm{in}}\right\|^2_{L^2\left(\langle v\rangle^{k_2}\,\mathrm{d}x\,\mathrm{d}\mu\right)}.$$

□

4.3.3 Proof of Proposition 4.3

Using the convolution $U \star V = \int_0^t U(t-s)V(s)\,\mathrm{d}s$, Duhamel's formula asserts that

$$e^{t(\mathsf{L}-\mathsf{T})} = e^{t\mathsf{B}} + e^{t\mathsf{B}} \star \mathsf{C}\, e^{t(\mathsf{L}-\mathsf{T})}.$$

By Lemma 4.5 and (4.15) with $k = k_1$, ℓ as in Lemma 4.4 and $k_2 > k + 2\,\ell$, we get that

$$\left\| e^{t(\mathsf{L}-\mathsf{T})} \right\|_{L^2\left(\langle v\rangle^{k_1}\,\mathrm{d}x\,\mathrm{d}\mu\right)\to L^2\left(\langle v\rangle^{k_1}\,\mathrm{d}x\,\mathrm{d}\mu\right)} \le 1 + a\,\langle R\rangle^{\frac{k_2}{2}} \int_0^t \frac{C\,\mathrm{d}s}{(1+s)^{\frac{k_2-k_1}{2\,\ell}}}$$

is bounded uniformly in time. □

4.4 Proof of Theorem 4.1

The control of the macroscopic part $\langle \mathsf{AT\Pi} f, \mathsf{\Pi} f\rangle$ is achieved exactly as in [44]. We sketch the proof for the sake of completeness.

Lemma 4.6. *Under the assumptions of Theorem 4.1, for any $f \in L^1(\mathrm{d}x\,\mathrm{d}v) \cap L^2(\mathrm{d}x\,\mathrm{d}\mu)$,*

$$\langle \mathsf{AT\Pi} f, \mathsf{\Pi} f\rangle \ge \Phi\big(\|\mathsf{\Pi} f\|^2\big) \quad \textit{with} \quad \Phi^{-1}(y) := 2\,y + \left(\frac{y}{\mathsf{c}}\right)^{\frac{d}{d+2}}, \quad \mathsf{c} = \Theta\, \mathcal{C}_{Nash}^{-\left(1+\frac{2}{d}\right)}\, \|f\|_{L^1(\mathrm{d}x\,\mathrm{d}v)}^{-\frac{4}{d}}.$$

Proof. With u defined by (4.9), we control $\|\mathsf{\Pi} f\|^2 = \|\rho_f\|^2_{L^2(\mathrm{d}x)}$ by $\langle \mathsf{AT\Pi} f, \mathsf{\Pi} f\rangle$ according to

$$\|\mathsf{\Pi} f\|^2 = \|u\|^2_{L^2(\mathrm{d}x)} + 2\,\Theta\,\|\nabla u\|^2_{L^2(\mathrm{d}x)} + \Theta^2\,\|\Delta u\|^2_{L^2(\mathrm{d}x)} \le \|u\|^2_{L^2(\mathrm{d}x)} + 2\,\langle \mathsf{AT\Pi} f, f\rangle.$$

using (4.10). Then we observe that, for any $t \ge 0$,

$$\|u(t,\cdot)\|_{L^1} = \|\rho_f(t,\cdot)\|_{L^1} = \|f\|_{L^1(\mathrm{d}x\,\mathrm{d}v)}\,, \quad \|\nabla u(t,\cdot)\|^2_{L^2(\mathrm{d}x)} \le \frac{1}{\Theta}\,\langle \mathsf{AT\Pi} f(t,\cdot,\cdot), f(t,\cdot,\cdot)\rangle$$

and use Nash's inequality

$$\|u\|^2_{L^2(\mathrm{d}x)} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac{4}{d+2}}_{L^1(\mathrm{d}x)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{L^2(\mathrm{d}x)}$$

to conclude the proof. □

The control of $(1-\mathsf{\Pi})f$ by the entropy production term relies on a new estimate.

Lemma 4.7. *Under the assumptions of Theorem 4.1, for any $f \in L^2\left(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\right) \cap L^1(\mathrm{d}x\,\mathrm{d}v)$,*

$$\|(1-\mathsf{\Pi})f\|^2_\beta \ge \Psi\left(\|(1-\mathsf{\Pi})f\|^2\right),$$

where $\mathcal{K}_k$ is as in Proposition 4.3, $\Psi(y) := C_0\,y^{1-\beta/k}$, $C_0 := \left(\mathcal{K}_k\left(1+\Theta_k\right)\|f\|_{L^2\left(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\right)}\right)^{2\frac{\beta}{k}}$.

Proof. Hölder's inequality

$$\|(1-\Pi)f\| \leq \|(1-\Pi)f\|_\beta^{\frac{k}{k-\beta}} \, \|(1-\Pi)f\|_{L^2\left(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}\mu\right)}^{\frac{|\beta|}{k-\beta}},$$

and

$$\begin{aligned}\|(1-\Pi)f\|_{L^2\left(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}\mu\right)} &\leq \|f\|_{L^2\left(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}\mu\right)} + \Theta_k \,\|\rho\|_{L^2(\,\mathrm{d}x)} \\ &\leq (1+\Theta_k)\,\|f\|_{L^2\left(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}\mu\right)} \leq \mathcal{K}_k\,(1+\Theta_k)\,\left\|f^{\mathrm{in}}\right\|_{L^2\left(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}\mu\right)},\end{aligned}$$

where the last inequality holds by Proposition 4.3, provide us with the estimate. □

Proof of Theorem 4.1. Using the estimates of Lemma 4.6 and Lemma 4.7, we obtain that

$$\|(1-\Pi)f\|_\beta^2 + \langle \mathsf{AT}\Pi f, \Pi f\rangle \geq \Psi\left(\|(1-\Pi)f\|^2\right) + \Phi\left(\|\Pi f\|^2\right).$$

Using (4.6), (4.7) and Proposition 4.2, we know that

$$\|(1-\Pi)f\|^2 \leq z_0 \quad \text{and} \quad \|\Pi f\|^2 \leq z_0 \quad \text{where} \quad z_0 := \frac{1+\delta}{1-\delta}\,\|f\|^2.$$

Thus, from

$$\Phi^{-1}(y) = y + \left(\frac{y}{\mathsf{c}}\right)^{\frac{d}{d+2}} \leq \left(C_1^{-1}\,y\right)^{\frac{d}{d+2}} \quad \text{with} \quad C_1 := \left(\Phi(z_0)^{\frac{2}{d+2}} + \mathsf{c}^{-\frac{d}{d+2}}\right)^{-\frac{d+2}{d}},$$

as long as $y \leq \Phi(z_0)$, we obtain

$$\Phi\left(\|\Pi f\|^2\right) \geq C_1\,\|\Pi f\|^{2\,\frac{d+2}{d}},$$

since $\|\Pi f\|^2 \leq z_0$. As a consequence,

$$\|(1-\Pi)f\|_\beta^2 + \langle \mathsf{AT}\Pi f, \Pi f\rangle \geq C_0\,\|(1-\Pi)f\|^{2\frac{k-\beta}{k}} + C_1\,\|\Pi f\|^{2\frac{d+2}{d}} \geq \min\left\{C_0, C_1\right\}\,\|f\|^{2+2/a},$$

where $a^{-1} = \max\{2/d, |\beta|/k\}$. Collecting terms, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}[f] \leq -\,C\,a\,\mathsf{H}[f]^{1+1/a},$$

using (4.6), (4.7) and

$$C := \frac{\kappa}{a}\,\min\left\{C_0, C_1\right\}\left(\tfrac{2}{1+\delta}\right)^{1+1/a}.$$

Then the proof of Theorem 4.1 follows from a Grönwall estimate. □

As a concluding remark, the observation that a control of the solution in the space $L^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$, based on Proposition 4.3, is enough to prove Theorem 4.1. This is new in L^2 hypocoercive methods, and consistent with the homogeneous case (see Appendix B.2).

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Chapter 5

Fractional Hypocoercivity

Abstract

This chapter is devoted to kinetic equations without confinement. We investigate the large time behavior induced by collision operators with fat tail local equilibria. Such operators have an anomalous diffusion limit. In the appropriate scale, the macroscopic equation involves a fractional diffusion operator so that the optimal decay rate is determined by a fractional Nash inequality. At kinetic level we develop an L^2 hypocoercivity approach and establish a rate of decay compatible with the anomalous diffusion limit.

Résumé

Ce chapitre s'intéresse à des équations cinétiques sans confinement. On étudie le comportement en temps long induit par des opérateurs de collision dont l'équilibre microscopique a une décroissance polynomiale. De tels opérateurs peuvent avoir une limite de diffusion anormale. Dans l'échelle appropriée, l'équation macroscopique implique un opérateur de diffusion fractionnaire de telle sorte que le taux de décroissance est déterminé par une inégalité de Nash fractionnaire. Au niveau cinétique, on développe une approche d'hypocoercivité dans L^2 et on établit un taux de décroissance compatible avec celui de la diffusion fractionnaire.

Contents

5.1 Introduction

5.1.1 The model

We consider the Cauchy problem

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = \mathsf{L} f \\ f(0,x,v) = f^{\mathrm{in}}(x,v), \end{cases} \tag{5.1}$$

for a distribution function $f(t,x,v)$, with *position* variable in the whole space, $x\in\mathbb{R}^d$, with *velocity* variable $v\in\mathbb{R}^d$, and with *time* $t\geq 0$. All collision operators share the properties that the null space of L is spanned by the *local equilibrium* F, and that L only acts on the velocity variable.

The aim of this work is to handle the situation where F has a polynomial decay:

$$\forall v\in\mathbb{R}^d,\qquad F(v) = \frac{C_\gamma}{\langle v\rangle^{d+\gamma}}, \tag{5.2}$$

with $\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}}$, $\gamma > 0$ and C_γ a normalisation constant ensuring that F is a probability density. We shall cover three cases of linear collision operators: the *Fokker-Planck* operator

$$\mathsf{L}_1 f = \nabla_v \cdot \Big(F \, \nabla_v \Big(F^{-1} f\Big)\Big), \tag{a}$$

the *linear Boltzmann* operator, or *scattering* collision operator

$$\mathsf{L}_2 f = \int_{\mathbb{R}^d} \mathrm{b}(\cdot, v') \left(f(v') \, F(\cdot) - f(\cdot) \, F(v') \right) \mathrm{d}v', \tag{b}$$

and the *fractional Fokker-Planck* operator

$$\mathsf{L}_3 f = \Delta_v^{\mathfrak{a}/2} f + \nabla_v \cdot (E \, f), \tag{c}$$

with $\mathfrak{a} \in (0, 2)$. In this latter case, we shall simply assume that the *friction force* E is radial and solves the equation

$$\Delta_v^{\mathfrak{a}/2} F + \nabla_v \cdot (E \, F) = 0. \tag{5.3}$$

As in [161], for the Linear Boltzmann operator, we have in mind a *collision kernel* b which take the form $\mathrm{b}(v, v') = \langle v' \rangle^\beta \langle v \rangle^\beta$ or $\mathrm{b}(v, v') = |v' - v|^\beta$ and in particular we shall assume that the *collision frequency* ν is positive, locally bounded and verifies

$$\nu(v) := \int_{\mathbb{R}^d} \mathrm{b}(v, v') F(v') \, \mathrm{d}v' \underset{v \to \infty}{\sim} |v|^\beta, \tag{H0}$$

for a given $\beta \in \mathbb{R}$. Inspired by the fractional diffusion limit proved in [161] and remarking that the above operators can formally be written on the form $\mathsf{B}(f) - \nu(v) \, f$, we define more generally β as the exponent of polynomial behavior at infinity of the formal function ν. We thus define β through

$$\beta = -2 \text{ in case (a), and } \beta = \gamma - \mathfrak{a} \text{ in case (c).}$$

The definition of β in the last case comes from the fact that under the condition (5.3), E verifies

$$E(v) = G(v) \, \langle v \rangle^\beta \, v, \tag{5.4}$$

where $G \in L^\infty(\mathbb{R}^d)$ is a positive function such that $G^{-1} \in L^\infty(B_0^c(1))$, which is proved in Proposition C.28 in appendix.

The collision kernel is also supposed to conserve the mass, which is equivalent to

$$\int_{\mathbb{R}^d} (\mathrm{b}(v, v') - \mathrm{b}(v', v)) \, F(v') \, \mathrm{d}v' = 0. \tag{H1}$$

We also need an assumption which will be used to guarantee some microscopic coercivity

$$\sup_{v \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{\nu(v)}{\mathrm{b}(v, v')} F' \mathrm{d}v' + \left(\int_{\mathbb{R}^d} \frac{\mathrm{b}(v, v')^2}{\nu(v') \nu(v)^2} F' \mathrm{d}v' \right)^{\frac{1}{2}} \right) < \infty. \tag{H2}$$

Moreover, for technical reasons we assume additionally that

$$\iint_{\mathbb{R}^{2d}} \frac{\mathrm{b}(v',v)^2}{\nu(v')\nu(v)} F F' \,\mathrm{d}v\,\mathrm{d}v' < \infty, \tag{H3}$$

and when $\beta < 0$

$$\forall k \in (0, \gamma-\beta), \int_{\mathbb{R}^d} \frac{\mathrm{b}(v',v)\,\langle v'\rangle^k}{\langle v\rangle^\beta} F' \,\mathrm{d}v' < \infty. \tag{H4}$$

All these assumptions are verified when

$$\begin{aligned} &\mathrm{b}(v',v) \le \langle v'\rangle^\beta \langle v\rangle^\beta && \text{with } \beta < \gamma \\ &\mathrm{b}(v',v) \le |v'-v|^\beta && \text{with } \beta \in (-d/2, \gamma) \end{aligned}$$

5.1.2 The link with fractional diffusion.

For any given $\alpha \in (0,2)$, consider the rescaled kinetic equation

$$\varepsilon^\alpha \partial_t f + \varepsilon v \cdot \nabla_x f = \mathsf{L} f. \tag{5.5}$$

We shall say that (5.5) has a *fractional diffusion limit* if the solution f_ε of (5.5) converges as $\varepsilon \to 0_+$ to a function $\rho(t,x)\,F(v)$ where ρ solves the *fractional heat equation*

$$\partial_t \rho = \kappa\,\Delta^{\alpha/2}\rho. \tag{5.6}$$

By computing $\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^2}^2 = -\,2\,\kappa\,\left\|\Delta^{\alpha/4}\rho\right\|_{L^2}^2$ and using the *fractional Nash inequality*

$$\|\rho\|_{L^2}^2 \le \mathcal{C}_{\mathrm{Nash}}^\alpha \left\|\Delta^{\alpha/4}\rho\right\|_{L^2}^{\frac{2d}{d+\alpha}} \|\rho\|_{L^1}^{\frac{2\alpha}{d+\alpha}}\,, \tag{5.7}$$

with optimal constant $\mathcal{C}_{\mathrm{Nash}}^\alpha$, we obtain that the solution of (5.6) decays according to

$$\forall\, t \ge 0\,,\quad \|\rho(t,\cdot)\|_{L^2}^2 \le \left(\|\rho(0,\cdot)\|_{L^2}^{-2\,\frac{\alpha}{d}} + 2\,\kappa\,\tfrac{\alpha}{d}\left(\mathcal{C}_{\mathrm{Nash}}^\alpha\right)^{-2\,(1+\frac{\alpha}{d})} \|\rho\|_{L^1}^{-2\,\frac{\alpha}{d}}\;t\right)^{-\frac{d}{\alpha}}.$$

So far $\|\rho\|_{L^p}$ denotes the $L^p(\mathbb{R}^d)$ norm. Our goal is to prove that a solution of (5.5) satisfies a similar bound, that is,

$$\forall\, t \ge 0\,,\quad \|f(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^{2d},\,F^{-1}\mathrm{d}x\,\mathrm{d}v)}^2 \le \mathcal{C}\,(1+t)^{-\frac{d}{\alpha}}\,, \tag{5.8}$$

for some positive constant $\mathcal{C}$.

In particular, the case when $\mathsf{L} = \mathsf{L}_2$ has been proved in [161]. In this paper, the authors obtain the value of α in the case when $\gamma > 0$, $\gamma > \beta$ and $\gamma + \beta < 2$ as

$$\alpha = \frac{\gamma - \beta}{1 - \beta}. \tag{5.9}$$

In the case $\gamma+\beta > 2$, the diffusion is classical, i.e. $\alpha = 2$, which had already been proved in [73]. In the case of the Fokker-Planck operator $\mathsf{L} = \mathsf{L}_1$, the result also holds by taking $\beta = -2$, which has been proved in dimension $d = 1$ in [141] and in all dimensions in [93]. The fact that $\beta = -2$ can be understood by remarking that the Fokker-Planck operator can be written

$$\begin{aligned} \mathsf{L}_1 f &= \Delta_v f + (d+\gamma)\,\mathrm{div}_v\Big(\langle v\rangle^{-2}\, v f\Big) \\ &= \Delta_v f + (d+\gamma)\,\langle v\rangle^{-2}\, v\cdot\nabla_v f + (d+\gamma)\Big(d - 2\tfrac{|v|^2}{\langle v\rangle^2}\Big)\langle v\rangle^{-2} f \end{aligned}$$

so that the term of order one behaves like $|v|^{-2}$ for large $|v|$. In the case when $\mathsf{L} = \mathsf{L}_3$, the diffusive limit has been proved for $E(v) = v$ in which case $F(v) \underset{v\to\infty}{\sim} \frac{C}{\langle v\rangle^{d+\mathfrak{a}}}$ and $\alpha = \mathfrak{a}$. However, we expect from Formula (5.4) that the same diffusion limit holds in this case with α defined by (5.9).

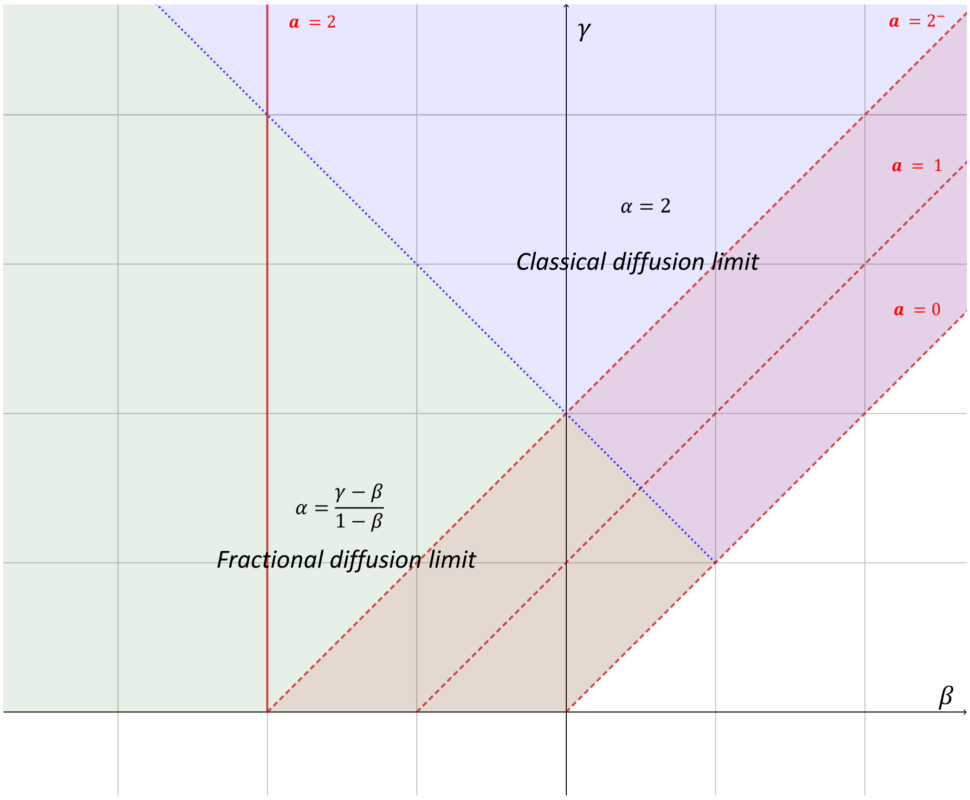

Figure 5.1: Diffusive limit depending on β and γ for the three operators. The green part corresponds to the case when $\alpha < 2$ and the blue part to the case when $\alpha = 2$. The red vertical line is the case when L can be taken as the Fokker-Planck operator whereas the red diagonal zone corresponds to the cases where it can be the fractional Fokker-Planck operator.

5.1.3 The hypocoercivity strategy

Let us consider the measure $\mathrm{d}\mu = F^{-1}(v)\,\mathrm{d}v$ and the Fourier transform of f in x defined by

$$\widehat{f}(t,\xi,v) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-\,i\,x\cdot\xi} f(t,x,v)\,\mathrm{d}x.$$

If f solves (5.5), then in the equation satisfied by $\widehat{f}$,

$$\partial_t\widehat{f} + \mathsf{T}\widehat{f} = \mathsf{L}\widehat{f}\,, \quad \widehat{f}(0,\xi,v) = \widehat{f}_0(\xi,v)\,, \quad \mathsf{T}\widehat{f} = i\,(v\cdot\xi)\widehat{f}\,, \tag{5.10}$$

$\xi \in \mathbb{R}^d$ can be seen as a parameter and for each Fourier mode ξ, one can study the decay of f. This is what we call the *mode-by-mode analysis*, as in [44]. Therefore, for any given $\xi \in \mathbb{R}^d$, we consider $g : (t,v) \mapsto \widehat{f}(t,\xi,v)$ on the complex valued Hilbert space $\mathcal{H} = L^2\,(\mathrm{d}\mu)$ with scalar product and norm given by

$$\langle f,g\rangle = \int_{\mathbb{R}^d} \bar{f} g\,\mathrm{d}\mu \quad\text{and}\quad \|f\| = \int_{\mathbb{R}^d} |f|^2\,\mathrm{d}\mu.$$

Let Π be the orthogonal projection on the subspace generated by F

$$\Pi g = \rho_g\,F \quad\text{where}\quad \rho_g := \int_{\mathbb{R}^d} g\,\mathrm{d}v\,.$$

Let us observe that the property $\Pi\mathsf{T}\Pi = 0$ holds as a consequence of the radial symmetry of F. Always in Fourier variable, we can define the entropy as follows. First we define

$$\varphi(\xi,v) := \frac{\langle v\rangle^{-\beta}}{1+\langle v\rangle^{2|1-\beta|}\,|\xi|^2} \quad\text{and}\quad \varphi_0(\xi,v) := \frac{1}{1+\langle v\rangle^2\,|\xi|^2}.$$

and we denote by ψ the normalization of φ_0 in the L^2 norm, which is defined by $\psi := \frac{\varphi_0}{\|\varphi_0(\xi,\cdot)\|_{L^2}}$. The entropy $\hat{\mathsf{H}}$ is then defined by

$$\begin{aligned} \hat{\mathsf{H}}(f) &:= \left\|\widehat{f}\right\|^2 + \delta\,\mathrm{Re}\left\langle \mathsf{A}_\xi\widehat{f},\widehat{f}\right\rangle \\ \mathsf{A}_\xi\widehat{f} &:= \psi\,\Pi\mathsf{T}\,\varphi\,\widehat{f}, \end{aligned} \tag{5.11}$$

for a given constant $\delta \in (0,1)$.

Remark 5.1. *By definition, the operator A_ξ can also be written more explicitly in an integral form as follow*

$$\mathsf{A}_\xi\widehat{f}(\xi,v) := \psi(\xi,v)\,F(v)\int_{\mathbb{R}^d}(-iv'\cdot\xi)\,\varphi(\xi,v')\,\widehat{f}(\xi,v')\,\mathrm{d}v'.$$

Remark 5.2. *Note that with all these definitions we obtain $\|\psi(\xi,\cdot)\|_{L^2} = 1$ and $|(v\cdot\xi)\varphi(\xi,v)| \le 1$, so that the Cauchy-Schwarz inequality yields*

$$|\langle\mathsf{A}_\xi g,g\rangle| \le \left|\int_{\mathbb{R}^d}\psi(\xi,v)g(\xi,v)\,\mathrm{d}v\right|\left|\int_{\mathbb{R}^d}(v\cdot\xi)\varphi(\xi,v)g(\xi,v)\,\mathrm{d}v\right| \le \|g\|^2\,.$$

Thus, $\hat{\mathsf{H}}$ is equivalent to the $L^2(\mathrm{d}\mu)$ norm and more precisely we get

$$(1-\delta)\,\|f\|^2 \le \hat{\mathsf{H}}(f) \le (1+\delta)\,\|f\|^2\,.$$

Remark 5.3. *One can compare this new entropy to the one introduced in [81] and used in previous works where the diffusion limit is not fractional [81, 44, 45]. In these works, the operator* A *was defined by*

$$\mathsf{A} = \left(1 + |\mathsf{T}\Pi|^2\right)^{-1} (\mathsf{T}\Pi)^*,$$

which can be written in Fourier variable as

$$\mathsf{A} = \Pi \frac{-iv \cdot \xi}{1 + \Theta|\xi|^2},$$

where $\Theta = \int_{\mathbb{R}^d} |v \cdot \mathbf{e}|^2 F(v) \, \mathrm{d}v$ *for an arbitrary unitary vector* $\mathbf{e}$*. One directly remarks that this operator is not defined if* F *does not have at least finite moments of order* 2*. The new operator* A_ξ *can be written*

$$\mathsf{A}_\xi = \frac{C_\xi}{1 + \langle v \rangle^2 |\xi|^2} \Pi \frac{(-iv \cdot \xi) \langle v \rangle^{-\beta}}{1 + \langle v \rangle^{2|1-\beta|} |\xi|^2}.$$

The function on the left of Π *was added to control the cases when* F *does not have* 2 *moments, and the apparition of the* β *in the operator provides the good scaling corresponding to the fractional diffusion limit in [161]. In this paper, the symbol of the limiting diffusion operator is indeed obtained as*

$$\int_{\mathbb{R}^d} a(\xi, v) F(v) \, \mathrm{d}v,$$

where

$$a(\xi) = \frac{\nu(v)}{\nu(v) - iv \cdot \xi} = \frac{\nu(v) \left(\nu(v) + iv \cdot \xi\right)}{\nu(v)^2 + |v \cdot \xi|^2},$$

and by assuming that $\nu(v) = \langle v \rangle^\beta$*, the leading order term is given by*

$$\frac{(iv \cdot \xi) \langle v \rangle^\beta}{\langle v \rangle^{2\beta} + |v \cdot \xi|^2} = \frac{(iv \cdot \xi) \langle v \rangle^{-\beta}}{1 + \langle v \rangle^{-2\beta} |v \cdot \xi|^2}.$$

The new operator A_ξ *can be seen as a mix between these two approaches.*

5.1.4 Decay rates

By obtaining the decay of the above defined entropy and using the fact that it is equivalent to the $L^2(\mathrm{d}x \, \mathrm{d}\mu)$ norm, we obtain the following results. The first one corresponds to the case when the diffusion limit is classical.

Theorem 5.4. *Assume* $\gamma > \beta$ *and* $\gamma + \beta > 2$ *and let* f *be a solution of* (5.1) *with initial condition* $f^{\mathrm{in}} \in L^1(\mathrm{d}x \, \mathrm{d}v) \cap L^2(\mathrm{d}x \, \mathrm{d}\mu)$*. Then the following alternative holds true.*

- *If* $\beta > 0$*, then it holds*

$$\|f\|^2_{L^2(\mathrm{d}x \, \mathrm{d}\mu)} \leq C \, \frac{\|f^{\mathrm{in}}\|^2_{L^1(\mathrm{d}x \, \mathrm{d}v)} + \|f^{\mathrm{in}}\|^2_{L^2(\mathrm{d}x \, \mathrm{d}\mu)}}{(1+t)^{\frac{d}{2}}}.$$

- *If $\beta \le 0$ and $f^{\rm in} \in L^2(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}\mu)$ for a given $k \in (0,\gamma)$, then it holds*

$$\|f\|^2_{L^2(\mathrm{d}x\,\mathrm{d}\mu)} \le C\,\frac{\|f^{\rm in}\|^2_{L^1(\mathrm{d}x\,\mathrm{d}v)} + \|f^{\rm in}\|^2_{L^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu)}}{(1+t)^{\min\left(\frac{d}{2},\frac{k}{|\beta|}\right)}}.$$

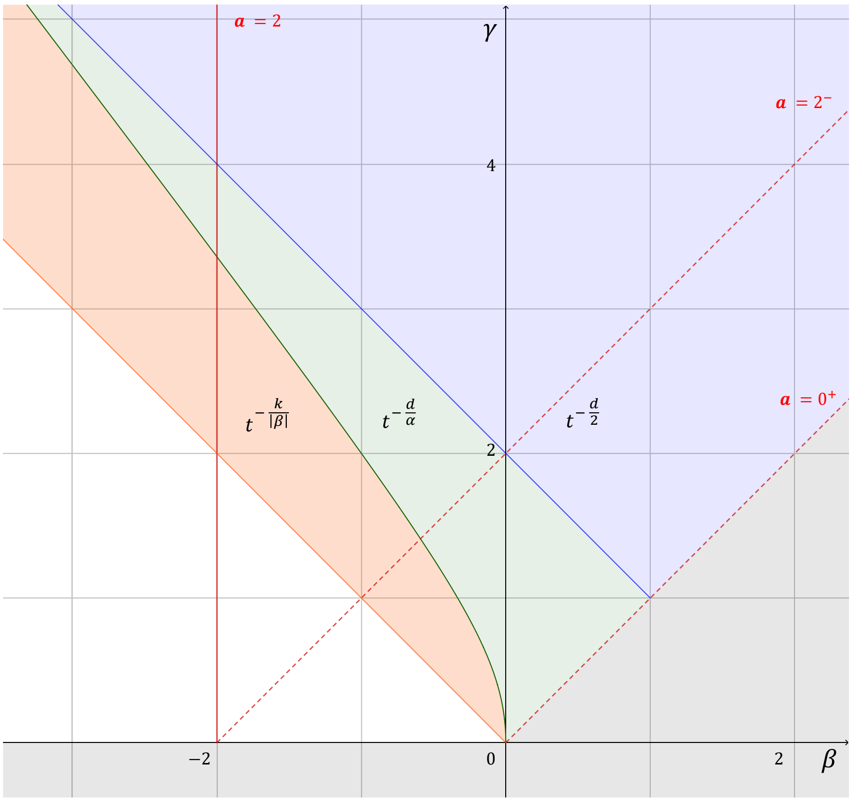

Figure 5.2: Decay rates of the main theorems. The blue part corresponds to the case when the decay is the decay of the heat equation, the green part to its fractional version, and the orange part is the case when the microscopic behavior is dominant.

Theorem 5.5. *Assume now $\gamma > |\beta|$ and $\gamma + \beta < 2$ and define*

$$\alpha = \frac{\gamma-\beta}{1-\beta} \in (0,2).$$

Again, let f be a solution of (5.1) with initial condition $f^{\rm in} \in L^1(\mathrm{d}x\,\mathrm{d}v) \cap L^2(\mathrm{d}x\,\mathrm{d}\mu)$. Then the following alternative holds true.

- *If $\beta > 0$, then it holds*

$$\|f\|^2_{L^2(\mathrm{d}x\,\mathrm{d}\mu)} \le C\,\frac{\|f^{\rm in}\|^2_{L^1(\mathrm{d}x\,\mathrm{d}v)} + \|f^{\rm in}\|^2_{L^2(\mathrm{d}x\,\mathrm{d}\mu)}}{(1+t)^{\frac{d}{\alpha}}}.$$

- *If $\beta \leq 0$ and $f^{\mathrm{in}} \in L^2(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}\mu)$ for a given $k \in (0,\gamma)$, then it holds*

$$\|f\|^2_{L^2(\mathrm{d}x\,\mathrm{d}\mu)} \leq C\,\frac{\|f^{\mathrm{in}}\|^2_{L^1(\mathrm{d}x\,\mathrm{d}v)} + \|f^{\mathrm{in}}\|^2_{L^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu)}}{(1+t)^{\min\left(\frac{d}{\alpha},\frac{k}{|\beta|}\right)}}.$$

The results are summarized in Figure 5.2.

5.1.5 Microscopic weighted coercivity

Contrary to [44], we consider here systems which do not always satisfy a microscopic coercivity assumption. However, the operators will at least satisfy a weaker condition of *microscopic weighted coercivity* which is given by

$$-\langle \mathsf{L}g, g\rangle \geq \mathcal{C}_m \,\|(1-\Pi)g\|^2_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)}\,. \tag{5.12}$$

In the case when $\gamma+\beta<0$, this inequality does not hold anymore, which implies that other methods should be used.

In case (a), $\beta=-2$ and the microscopic condition is given by the following weighted Poincaré inequality for $\gamma>2$

$$\int_{\mathbb{R}^d} |h-\langle h\rangle_F|\,\langle v\rangle^{-2}\,F\,\mathrm{d}v \leq \mathcal{C}_P \int_{\mathbb{R}^d} |\nabla_v h|^2 F\,\mathrm{d}v, \tag{5.13}$$

where $\langle h\rangle_F = \int_{\mathbb{R}^d} hF\mathrm{d}v$ and which is sometimes called Lyapunov-Poincaré or weak Poincaré inequality (see e.g. [16, 130]) and is similar but different from the Hardy-Poincaré inequality in which the optimal constant is known (see e.g. [38, 39, 83]).

In case (b), this is a consequence of the following inequality (see [161])

$$\int_{\mathbb{R}^d} |h-\langle h\rangle_F|^2\,\nu F\,\mathrm{d}v \leq \mathcal{C}_P \iint_{\mathbb{R}^{2d}} \mathrm{b}(v,v')|h(v')-h(v)|^2 FF'\,\mathrm{d}v\,\mathrm{d}v'$$

where

$$\mathcal{C}_P \leq \sup_{v\in\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\frac{\nu(v)}{\mathrm{b}(v,v')}F'\mathrm{d}v' + \left(\int_{\mathbb{R}^d}\frac{\mathrm{b}(v,v')^2}{\nu(v')\nu(v)^2}F'\mathrm{d}v'\right)^{\frac{1}{2}}\right).$$

In case (c), this is a consequence of the following weighted fractional Poincaré inequality which is a corollary of [70, Theorem 4.3]

$$\int_{\mathbb{R}^d} |h-\langle h\rangle_F|^2\,\langle v\rangle^\beta\,F\,\mathrm{d}v \leq \mathcal{C}_P \iint_{\mathbb{R}^{2d}} \frac{|h(v')-h(v)|^2}{|v'-v|^{d+\mathfrak{a}}}F\,\mathrm{d}v\,\mathrm{d}v',$$

which is true if and only if $\gamma > \frac{\mathfrak{a}}{2}$, or equivalently $\gamma+\beta>0$.

5.1.6 A brief review of the literature

Fractional diffusion limits of kinetic equations attracted a considerable interest in the recent years. The microscopic jump processes are indeed easy to encode in kinetic equations and the diffusion limit provides a simple procedure to justify the use of fractional operators at macroscopic level. We refer to [22, 161] for an introduction to the topic in the case of operators of scattering type and a discussion of earlier results on standard, *i.e.*, non-fractional, diffusion limits. In a recent paper, [141], the case of a Fokker-Planck equation with heavy tails local equilibria has been considered (only when $d = 1$) and, in from a probabilistic point of view, [94] collects various related results. The diffusion limit of the fractional Fokker-Planck equation has been studied in [2].

In the homogeneous case, that is, when there is no x-dependence, it is classical to introduce a potential function $\Phi(v) = -\log F(v)$ and classify the behavior of the solution h to (5.1) according to the growth rate of Φ. Assume that the collision operator is either the Fokker-Planck operator of case (a) or the scattering operator of Case (b). Schematically, if

$$\Phi(v) = \left(1 + |v|^2\right)^{\gamma/2},$$

we obtain that $\|h(t,\cdot) - F\|_{L^2(\mathbb{R}^d,\mathrm{d}\mu)}$ decays exponentially if $\gamma \geq 1$. Here we assume that $\|h(t,\cdot)\|_{L^1(\mathbb{R}^d)} = \|F\|_{L^1(\mathbb{R}^d)}$. In the range $\gamma \in (0,1)$, the Poincaré inequality of Case (a) has to be replaced by a *weak Poincaré* or a *weighted Poincaré inequality*: see [184, 130] and rates of convergence are typically algebraic in t. Summarizing, the lowest is the rate of growth as $|x| \to +\infty$, the slowest is the rate of convergence of h to F. The turning point occurs for the minimal growth which guarantees that F is integrable, at least for solutions of the homogeneous equation with initial data in $L^1(\mathbb{R}^d)$. If we consider for instance

$$\Phi(v) = \tfrac{\gamma}{2} \log\left(1 + |v|^2\right),$$

with $\gamma < d$, then diffusive effects win over confinement and the unique stationary solution with finite mass is 0. To measure the sharp rate of decay of h towards 0, one can replace the Poincaré inequality and the weak Poincaré inequality by *Nash's inequality*. See [45].

Standard diffusion limits provide an interesting insight into the *micro/macro decomposition* which has been the key of the L^2 *hypocoercive approach* of [81]. Another parameter has now to be taken into account: the confinement in the spatial variable x. In presence of a confining potential $V = V(x)$ with sufficient growth and when F has a fast decay rate, typically a quadratic growth, the rate of convergence is found to be exponential. A milder growth of V gives a slower convergence rate as analyzed in [57]. With e^{-V} not integrable, diffusion again wins in the hypocoercive picture, and the rate of convergence of a finite mass solution of (5.1) towards 0 can be captured by Nash and related Caffarelli-Kohn-Nirenberg inequalities: see [44, 45].

A typical regime for fractional diffusion limits is given by local equilibria with fat tails which behave according to (5.2) with $\alpha \in (0,2)$: F is integrable but has no second moment. Whenever fractional diffusion limits can be obtained, we claim that rates of convergence can also be produced in an adapted L^2 hypocoercive approach. To simplify the exposition, we shall consider only the case $V = 0$ and measure the rate of convergence to 0. It is natural to expect that a fractional Nash type approach has to play the central role, and this is indeed what happens. The mode-by-mode hypocoercivity estimate shows that rates are of the order of $|\xi|^\alpha$ as $\xi \to 0$ which results in the expected time decay. In this direction, let us mention that the spectral information associated with $|\xi|^\alpha$ is very natural in connection with the fractional heat equation (5.6) as was recently shown in [23].

5.2 Mode by mode hypocoercivity

In this section, we show the first step towards the proof of our main theorems, that is the basic energy estimate coming from the hypocoercivity functional. To help the readability, we recall that

$$\mathsf{A}_\xi \widehat{f} = \psi(\xi,\cdot)\, F \int_{\mathbb{R}^d} (-iv\cdot\xi)\, \varphi(\xi,v)\, \widehat{f}(\xi,v)\,\mathrm{d}v.$$

Proposition 5.6. *Assume $\gamma > |\beta|$. Define*

$$\mu_{\mathsf{L}}(\xi) := \left\|\mathsf{L}^*\left((v\cdot\xi)\varphi(\xi,\cdot)F\right)\right\|_{L^2(\langle v\rangle^{-\beta}\mathrm{d}\mu)}, \qquad \lambda_{\mathsf{L}}(\xi) := \left\|\mathsf{L}^*\left(\psi(\xi,\cdot)F\right)\right\|_{L^2(\langle v\rangle^{-\beta}\mathrm{d}\mu)}.$$

Then there exists $\delta > 0$ such that if $\hat{\mathsf{H}}$ is defined by formula (5.11), it holds

$$\partial_t \hat{\mathsf{H}}(f) \le -C\delta \frac{|\xi|^{\alpha'}}{1+|\xi|^{\alpha'}} \left\|\Pi \widehat{f}\right\|^2 - \frac{1}{2}\mathcal{C}_P \left\|(1-\Pi)\widehat{f}\right\|^2_{L^2(\langle v\rangle^{\beta}\mathrm{d}\mu)},$$

where $\alpha' = 2 + \frac{\min(\gamma+\beta-2,0)}{|1-\beta|}$ (i.e. $\alpha' = \alpha$ if $\gamma+\beta < 2$ and $\alpha' = 2$ else).

Proof of Proposition 5.6. By the microscopic weighted coercivity (5.12), one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\widehat{f}\right\|^2 \le -\mathcal{C}_P \left\|(1-\Pi)\widehat{f}\right\|^2_{L^2(\langle v\rangle^{\beta}\mathrm{d}\mu)}.$$

Now we write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \mathsf{A}_\xi \widehat{f}, \widehat{f}\right\rangle &= -\left\langle \mathsf{A}_\xi \mathsf{T}\widehat{f}, \widehat{f}\right\rangle - \left\langle \mathsf{A}_\xi \widehat{f}, \mathsf{T}\widehat{f}\right\rangle + \left\langle \mathsf{A}_\xi \mathsf{L}\widehat{f}, \widehat{f}\right\rangle + \left\langle \mathsf{A}_\xi \widehat{f}, \mathsf{L}\widehat{f}\right\rangle \\ &:= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

- ***Step 1: a bound for** $I_1 = -\left\langle \mathsf{A}_\xi \mathsf{T}\widehat{f}, \widehat{f}\right\rangle$.*

  One can rewrite I_1 as

$$I_1 = -\left(\int_{\mathbb{R}^d} \widehat{f}(v)\psi(v,\xi)\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d} \varphi(v',\xi)|v'\cdot\xi|^2\,\overline{\widehat{f}(v')}\,\mathrm{d}v'\right).$$

Using the micro-macro decomposition, we may decompose

$$\begin{aligned}
I_1 = &-\left(\int_{\mathbb{R}^d} \psi \Pi \widehat{f}\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d} \varphi(v)|v\cdot\xi|^2\,\overline{\Pi \widehat{f}(v)}\,\mathrm{d}v\right)\\
&-\left(\int_{\mathbb{R}^d} \psi (1-\Pi) \widehat{f}\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d} \varphi(v)|v\cdot\xi|^2\,\overline{\Pi \widehat{f}(v)}\,\mathrm{d}v\right)\\
&-\left(\int_{\mathbb{R}^d} \psi \Pi \widehat{f}\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d} \varphi(v)|v\cdot\xi|^2\,\overline{(1-\Pi) \widehat{f}(v)}\,\mathrm{d}v\right)\\
&-\left(\int_{\mathbb{R}^d} \psi (1-\Pi) \widehat{f}\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d} \varphi(v)|v\cdot\xi|^2\,\overline{(1-\Pi) \widehat{f}(v)}\,\mathrm{d}v\right),
\end{aligned}$$

so that

$$\begin{aligned}
I_1 = &-\left\|\Pi \widehat{f}\right\|^2 \|\psi F\|_{L^1(\mathrm{d}v)}\left(\int_{\mathbb{R}^d} \varphi(v)|v\cdot\xi|^2 F(v)\,\mathrm{d}v\right)\\
&-\overline{\rho_{\widehat{f}}}\left(\int_{\mathbb{R}^d} \psi (1-\Pi) \widehat{f}\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d} \varphi(v)|v\cdot\xi|^2 F(v)\,\mathrm{d}v\right)\\
&-\rho_{\widehat{f}}\|\psi F\|_{L^1(\mathrm{d}v)}\left(\int_{\mathbb{R}^d} \varphi(v)|v\cdot\xi|^2\,\overline{(1-\Pi) \widehat{f}}\,\mathrm{d}v\right)\\
&-\left(\int_{\mathbb{R}^d} \psi (1-\Pi) \widehat{f}\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d} \varphi(v)|v\cdot\xi|^2\,\overline{(1-\Pi) \widehat{f}}\,\mathrm{d}v\right).
\end{aligned}$$

Then, by defining

$$\lambda_k := \left\||v\cdot\xi|^k \psi F\right\|_{L^1(\mathrm{d}v)} \quad \text{and} \quad \mu_k := \left\||v\cdot\xi|^k \varphi F\right\|_{L^1(\mathrm{d}v)},$$

one may estimate $\mathrm{Re}(I_1)$

$$\begin{aligned}
\mathrm{Re}(I_1) \le -\lambda_0\mu_2 \left\|\Pi \widehat{f}\right\|^2 &+ \mu_2 \left\|\Pi \widehat{f}\right\| \left|\int_{\mathbb{R}^d} \psi (1-\Pi)\widehat{f}\,\mathrm{d}v\right|\\
&+ \lambda_0 \left\|\Pi \widehat{f}\right\| \left|\int_{\mathbb{R}^d} \varphi(v)|v\cdot\xi|^2 (1-\Pi)\widehat{f}\,\mathrm{d}v\right|\\
&+ \left|\int_{\mathbb{R}^d} \psi (1-\Pi)\widehat{f}\,\mathrm{d}v\right| \left|\int_{\mathbb{R}^d} \varphi(v)|v\cdot\xi|^2 (1-\Pi)\widehat{f}\,\mathrm{d}v\right|.
\end{aligned}$$

Then, by Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\mathrm{Re}(I_1) \le -\lambda_0\mu_2 \left\|\Pi \widehat{f}\right\|^2 &+ \left(\tilde{\lambda}_0\mu_2 + \lambda_0\tilde{\mu}_2\right) \left\|\Pi \widehat{f}\right\| \left\|(1-\Pi)\widehat{f}\right\|_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)}\\
&+ \tilde{\lambda}_0\tilde{\mu}_2 \left\|(1-\Pi)\widehat{f}\right\|^2_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)},
\end{aligned}$$

where

$$\tilde{\lambda}_k := \left\||v\cdot\xi|^k \psi F\right\|_{L^2(\langle v\rangle^{-\beta} \mathrm{d}\mu)} \quad \text{and} \quad \tilde{\mu}_k := \left\||v\cdot\xi|^k \varphi F\right\|_{L^2(\langle v\rangle^{-\beta} \mathrm{d}\mu)}.$$

- ***Step 2: a bound for*** $I_2 = -\left\langle \mathsf{A}_\xi \widehat{f}, \mathsf{T}\widehat{f}\right\rangle$.

  Note that since φ and ψ commute with T and $\Pi\mathsf{T}\Pi = 0$, we get

$$\mathsf{A}_\xi \Pi = \psi \Pi\mathsf{T}\Pi \varphi = 0, \qquad \mathsf{A}_\xi^* \mathsf{T}\Pi = -\varphi \mathsf{T}\Pi\mathsf{T}\Pi\psi = 0.$$

This yields

$$\begin{aligned}
I_2 &= -\left\langle \mathsf{A}_\xi(1-\Pi)\widehat{f}, \mathsf{T}(1-\Pi)\widehat{f}\right\rangle \\
&= \left(\int_{\mathbb{R}^d} \varphi(v)(v\cdot\xi)\,\overline{(1-\Pi)\widehat{f}}\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d} \psi(v)(v\cdot\xi)(1-\Pi)\widehat{f}\,\mathrm{d}v\right).
\end{aligned}$$

Again, by Using the Cauchy-Schwarz inequality, we obtain

$$I_2 \leq \tilde{\lambda}_1\tilde{\mu}_1 \left\|(1-\Pi)\widehat{f}\right\|^2_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)}.$$

- ***Step 3: a bound for*** $I_3 = \left\langle \mathsf{A}_\xi \mathsf{L}\widehat{f}, \widehat{f}\right\rangle$**.**
  Since $\mathsf{L}\Pi = 0$, it holds

$$\begin{aligned}
\left\langle \mathsf{A}_\xi \mathsf{L}\widehat{f}, \widehat{f}\right\rangle &= \left\langle \mathsf{A}_\xi \mathsf{L}(1-\Pi)\widehat{f}, \widehat{f}\right\rangle \\
&= \int_{\mathbb{R}^d} \psi(\xi,\cdot)\int_{\mathbb{R}^d}(iv'\cdot\xi)\varphi(\xi,v')\mathsf{L}\overline{(1-\Pi)\widehat{f}}(v')\,\mathrm{d}v'\,\widehat{f}(v)\,\mathrm{d}v \\
&= \left(\int_{\mathbb{R}^d}\psi\widehat{f}\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^d}\mathsf{L}^*\left((v\cdot\xi)\varphi F\right)\,\overline{(1-\Pi)\widehat{f}}\,\mathrm{d}\mu\right).
\end{aligned}$$

With a micro-macro decomposition, one may estimate

$$\int_{\mathbb{R}^d}\psi\widehat{f}\,\mathrm{d}v = \lambda_0\rho_{\widehat{f}} + \int_{\mathbb{R}^d}\psi(1-\Pi)\widehat{f}\,\mathrm{d}v.$$

The Cauchy-Schwarz inequality gives

$$\left|\int_{\mathbb{R}^d}\mathsf{L}^*\left((v\cdot\xi)\varphi F\right)\,\overline{(1-\Pi)\widehat{f}}\mathrm{d}\mu\right| = \mu_{\mathsf{L}}\left\|(1-\Pi)\widehat{f}\right\|_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)}.$$

It yields

$$\begin{aligned}
\left|\left\langle \mathsf{A}_\xi \mathsf{L}\widehat{f}, \widehat{f}\right\rangle\right| &\leq (\lambda_0\left\|\Pi\widehat{f}\right\| + \tilde{\lambda}_0\left\|(1-\Pi)\widehat{f}\right\|_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)})\mu_{\mathsf{L}}\left\|(1-\Pi)\widehat{f}\right\|_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)} \\
&\leq \lambda_0\mu_{\mathsf{L}}\left\|\Pi\widehat{f}\right\|\left\|(1-\Pi)\widehat{f}\right\|_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)} + \tilde{\lambda}_0\mu_{\mathsf{L}}\left\|(1-\Pi)\widehat{f}\right\|^2_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)}.
\end{aligned}$$

- ***Step 4: a bound for*** $I_4 = \left\langle \mathsf{A}_\xi\widehat{f}, \mathsf{L}\widehat{f}\right\rangle$**.**
  Again, using the fact that $\mathsf{L}\Pi = 0$, $\mathsf{A}_\xi\Pi = 0$,

$$\begin{aligned}
\left|\left\langle \mathsf{A}_\xi\widehat{f}, \mathsf{L}\widehat{f}\right\rangle\right| &= \left|\left\langle \mathsf{L}^*F\psi(\xi,\cdot)\int_{\mathbb{R}^d}(-iv'\cdot\xi)\varphi(\xi,v')(1-\Pi)\widehat{f}(\xi,v')\,\mathrm{d}v', (1-\Pi)\widehat{f}\right\rangle\right| \\
&= \left|\int_{\mathbb{R}^d}(-iv'\cdot\xi)\varphi(\xi,v')F(v')(1-\Pi)\widehat{f}(\xi,v')\,\mathrm{d}\mu\right|\left|\left\langle \mathsf{L}^*F\psi(\xi,\cdot), (1-\Pi)\widehat{f}\right\rangle\right| \\
&\leq \lambda_{\mathsf{L}}\tilde{\mu}_1\left\|(1-\Pi)\widehat{f}\right\|^2_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)}.
\end{aligned}$$

the last step being obtained using Cauchy-Schwarz's inequality twice.

• ***Step 5: the differential inequality.***
Gathering all the estimates, we deduce

$$\partial_t \hat{\mathsf{H}}(f) \leq \delta \Big(-\lambda_0\mu_2 \left\| \Pi \widehat{f} \right\|^2 + \left(\tilde{\lambda}_0\mu_2 + \lambda_0\tilde{\mu}_2 + \lambda_0\mu_{\mathsf{L}} \right) \left\| \Pi \widehat{f} \right\| \left\| (1-\Pi)\widehat{f} \right\|_{L^2(\langle v \rangle^\beta \mathrm{d}\mu)} + (\tilde{\lambda}_0\tilde{\mu}_2 + \tilde{\lambda}_1\tilde{\mu}_1 + \tilde{\lambda}_0\mu_{\mathsf{L}} + \lambda_{\mathsf{L}}\tilde{\mu}_1) \left\| (1-\Pi)\widehat{f} \right\|^2_{L^2(\langle v \rangle^\beta \mathrm{d}\mu)} \Big) - \mathcal{C}_P \left\| (1-\Pi)\widehat{f} \right\|^2_{L^2(\langle v \rangle^\beta \mathrm{d}\mu)},$$

which by Young's inequality leads to

$$\partial_t \hat{\mathsf{H}}(f) \leq \delta \left(\frac{\sigma}{2} \left(\tilde{\lambda}_0\mu_2 + \lambda_0\tilde{\mu}_2 + \lambda_0\mu_{\mathsf{L}} \right) - \lambda_0\mu_2 \right) X^2 + \left(\delta \left(\tilde{\lambda}_0\tilde{\mu}_2 + \tilde{\lambda}_1\tilde{\mu}_1 + \tilde{\lambda}_0\mu_{\mathsf{L}} + \lambda_{\mathsf{L}}\tilde{\mu}_1 + \frac{1}{2\sigma} \left(\tilde{\lambda}_0\mu_2 + \lambda_0\tilde{\mu}_2 + \lambda_0\mu_{\mathsf{L}} \right) \right) - \mathcal{C}_P \right) Y^2.$$

One may choose $\sigma = \frac{\lambda_0\mu_2}{\tilde{\lambda}_0\mu_2 + \lambda_0\tilde{\mu}_2 + \lambda_0\mu_{\mathsf{L}}}$ to get

$$\partial_t \hat{\mathsf{H}}(f) \leq -\frac{\delta}{2}\lambda_0\mu_2 \left\| \Pi \widehat{f} \right\|^2 + (\delta\mathcal{C}(\xi) - \mathcal{C}_P) \left\| (1-\Pi)\widehat{f} \right\|^2_{L^2(\langle v \rangle^\beta \mathrm{d}\mu)},$$

with

$$\begin{aligned} \mathcal{C}(\xi) &= \tilde{\lambda}_0\tilde{\mu}_2 + \tilde{\lambda}_1\tilde{\mu}_1 + \tilde{\lambda}_0\mu_{\mathsf{L}} + \lambda_{\mathsf{L}}\tilde{\mu}_1 + \frac{1}{2\sigma} \left(\tilde{\lambda}_0\mu_2 + \lambda_0\tilde{\mu}_2 + \lambda_0\mu_{\mathsf{L}} \right) \\ &= \tilde{\lambda}_0\tilde{\mu}_2 + \tilde{\lambda}_1\tilde{\mu}_1 + \tilde{\lambda}_0\mu_{\mathsf{L}} + \lambda_{\mathsf{L}}\tilde{\mu}_1 + \frac{\left(\tilde{\lambda}_0\mu_2 + \lambda_0\tilde{\mu}_2 + \lambda_0\mu_{\mathsf{L}} \right)^2}{2\lambda_0\mu_2}. \end{aligned}$$

To infer that $\mathcal{C}(\xi)$ is in fact bounded in ξ, we need to estimate carefully various coefficients. This needs the following lemmas.

Lemma 5.7. *The coefficient μ_2, defined by*

$$\mu_2(\xi) = \int_{\mathbb{R}^d} \frac{|v\cdot\xi|^2 \langle v \rangle^{-\beta}}{1 + \langle v \rangle^{2|1-\beta|} |\xi|^2} \frac{c_\gamma \mathrm{d}v}{\langle v \rangle^{d+\gamma}},$$

is bounded when $|\xi|$ is large and satisfies

$$\mu_2(\xi) \underset{|\xi|\to 0}{\sim} C|\xi|^{\min\left(2, 2+\frac{\gamma+\beta-2}{|1-\beta|}\right)} \text{ if } \gamma+\beta \neq 2.$$

In the case when $\gamma+\beta = 2$, one obtains a logarithmic correction. The coefficients $\tilde{\mu}_1$ and $\tilde{\mu}_2$, that have been respectively defined by taking $k=1$ and $k=2$ in

$$\tilde{\mu}_k(\xi) = \left(\int_{\mathbb{R}^d} \left(\frac{|v\cdot\xi|^k \langle v \rangle^{-\beta}}{1 + \langle v \rangle^{2|1-\beta|} |\xi|^2} \right)^2 \frac{c_\gamma \mathrm{d}v}{\langle v \rangle^{d+\gamma+\beta}} \right)^{\frac{1}{2}},$$

satisfy

$$\tilde{\mu}_k(\xi) \underset{|\xi|\to 0}{\sim} C|\xi|^{\min\left(k, k+\frac{\gamma-2k+3\beta}{2|1-\beta|}\right)}, \qquad \tilde{\mu}_k(\xi) \underset{|\xi|\to\infty}{\sim} C|\xi|^{k-2},$$

when $\gamma + 3\beta \neq 2k$.

Lemma 5.8. *The coefficients λ_0 and λ_1 have been respectively defined by taking $k=0$ and $k=1$ in*

$$\lambda_k(\xi) := \left(\int_{\mathbb{R}^d} \frac{F(v)\mathrm{d}v}{(1+\langle v\rangle^2|\xi|^2)^2}\right)^{-\frac{1}{2}} \int_{\mathbb{R}^d} \frac{|v\cdot\xi|^k}{1+\langle v\rangle^2|\xi|^2}\frac{c_\gamma \mathrm{d}v}{\langle v\rangle^{d+\gamma}}.$$

For them, it holds

$$\lambda_k(\xi) \underset{|\xi|\to 0}{\sim} C|\xi|^{\min(k,\gamma)}, \text{ if } k\neq\gamma,$$
$$\lambda_k(\xi) \underset{|\xi|\to\infty}{\sim} |\xi|^k.$$

The coefficients $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$, that have been respectively defined by taking $k=0$ and $k=1$ in

$$\tilde{\lambda}_k(\xi) := \left(\int_{\mathbb{R}^d} \frac{F(v)\mathrm{d}v}{(1+\langle v\rangle^2|\xi|^2)^2}\right)^{-\frac{1}{2}} \left(\int_{\mathbb{R}^d} \left(\frac{|v\cdot\xi|^k}{1+\langle v\rangle^2|\xi|^2}\right)^2 \frac{\langle v\rangle^{-\beta} c_\gamma \mathrm{d}v}{\langle v\rangle^{d+\gamma}}\right)^{\frac{1}{2}},$$

satisfy

$$\tilde{\lambda}_k(\xi) \underset{|\xi|\to 0}{\sim} C|\xi|^{\min(k,\frac{\gamma+\beta}{2})}, \text{ if } k\neq\tfrac{\gamma+\beta}{2}$$
$$\tilde{\lambda}_k(\xi) \underset{|\xi|\to\infty}{\sim} |\xi|^k.$$

We now present the proof of Lemma 5.7. We omit the proof of Lemma 5.8 since it follows exactly the same steps.

Proof of Lemma 5.7. Start with the first claim. We first obtain the equivalence when $\xi\to 0$. Note that if $\gamma+\beta>2$, then

$$\mu_2(\xi) = \int_{\mathbb{R}^d} \frac{|v\cdot\xi|^2\langle v\rangle^{-\beta}}{1+\langle v\rangle^{2|1-\beta|}|\xi|^2}\frac{c_\gamma \mathrm{d}v}{\langle v\rangle^{d+\gamma}} \underset{\xi\to 0}{\sim} |\xi|^2\int_{\mathbb{R}^d}\frac{c_\gamma|v_1|^2}{\langle v\rangle^{d+\gamma+\beta}}\mathrm{d}v.$$

Now, if $\gamma+\beta<2$, we use the change of variable $v=|\xi|^{\frac{-1}{|1-\beta|}}u$ in the integral defining μ_2 to get

$$\mu_2(\xi) = \int_{\mathbb{R}^d} \frac{||\xi|^{\frac{|1-\beta|-1}{|1-\beta|}}u_1|^2\left\langle|\xi|^{\frac{-1}{|1-\beta|}}u\right\rangle^{-\beta}}{1+\left\langle|\xi|^{\frac{-1}{|1-\beta|}}u\right\rangle^{2|1-\beta|}|\xi|^2}\frac{|\xi|^{\frac{-d}{|1-\beta|}}c_\gamma \mathrm{d}u}{\left\langle|\xi|^{\frac{-1}{|1-\beta|}}u\right\rangle^{d+\gamma}}.$$

Note that for any $u\in\mathbb{R}^d\backslash\{0\}$, we have

$$\left\langle|\xi|^{\frac{-1}{|1-\beta|}}u\right\rangle^{-\beta} \underset{\xi\to 0}{\sim} |u|^{-\beta}|\xi|^{\frac{\beta}{|1-\beta|}}, \qquad 1+\left\langle|\xi|^{\frac{-1}{|1-\beta|}}u\right\rangle^{2|1-\beta|}|\xi|^2 \underset{\xi\to 0}{\sim} 1+|u|^{2|1-\beta|},$$
$$\left\langle|\xi|^{\frac{-1}{|1-\beta|}}u\right\rangle^{d+\gamma}|\xi|^{\frac{d}{|1-\beta|}} \underset{\xi\to 0}{\sim} |u|^{d+\gamma}|\xi|^{\frac{-\gamma}{|1-\beta|}}.$$

As a consequence,

$$\mu_2(\xi) \underset{\xi\to 0}{\sim} |\xi|^{2+\frac{\gamma+\beta-2}{|1-\beta|}} \int_{\mathbb{R}^d} \frac{|u_1|^2}{1+|u|^{2|1-\beta|}} \frac{c_\gamma \mathrm{d}u}{|u|^{d+\gamma+\beta}}.$$

Note that the later integral exists, since $\gamma+\beta-2>0$ and $\gamma\geq\beta$ (which implies $\gamma\geq 1$ when $\beta\geq 1$). Now turn to the equivalence when $\xi\to+\infty$. Note that if $\gamma-\beta>0$, then

$$\mu_2(\xi) = \int_{\mathbb{R}^d} \frac{|v\cdot\xi|^2 \langle v\rangle^\beta}{\langle v\rangle^{2\beta}+\langle v\rangle^2 |\xi|^2} \frac{c_\gamma \mathrm{d}v}{\langle v\rangle^{d+\gamma}} \underset{\xi\to 0}{\sim} \int_{\mathbb{R}^d} \frac{c_\gamma |v_1|^2}{\langle v\rangle^{d+\gamma+2-\beta}}\,\mathrm{d}v.$$

The first claim on μ_2 is now completed. The end of the lemma follows the same techniques, we omit the proof. □

These lemmas in hand, we are now ready to estimate $\mathcal{C}(\xi)$. Indeed, we get as a byproduct that λ_0, $\tilde\lambda_0$ are bounded away from zero. Thus,

$$\mathcal{C}(\xi) \lesssim \mu_{\mathsf{L}} + \lambda_{\mathsf{L}}\tilde\mu_1 + \frac{\mu_{\mathsf{L}}^2}{\mu_2} + \tilde\mu_2 + \tilde\lambda_1\tilde\mu_1 + \mu_2 + \frac{\tilde\mu_2^2}{\mu_2}.$$

Using all lemmas above, we conclude that $\mathcal{C}(\xi)$ is bounded. Now taking δ sufficiently small, we get that for all $\xi\in\mathbb{R}^n$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathsf{H}}(f) \leq -\frac{\delta}{2}\left(\mu_2(\xi)\left\|\Pi\hat f\right\|^2 + \left\|(1-\Pi)\hat f\right\|^2_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)}\right).$$

Recalling Lemma 5.7, we can bound μ_2 writing,

$$\mu_2(\xi) \gtrsim \frac{|\xi|^{2+\frac{\min(\gamma+\beta-2,0)}{|1-\beta|}}}{1+|\xi|^{2+\frac{\min(\gamma+\beta-2,0)}{|1-\beta|}}} = \frac{|\xi|^{\alpha'}}{1+|\xi|^{\alpha'}},$$

and the proposition follows. □

5.3 Estimates in weighted L^2 spaces

In this section, we assume that $\beta\leq 0$. We show the propagation of weighted norms $L^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu)$ with power law of order $k\in(0,\gamma)$. This will be crucially used to prove our main Theorem 5.5. The result is the following.

Proposition 5.9. *Let $k\in(0,\gamma)$ and f be solution of equation (5.1) with initial condition $f_{\mathrm{in}}\in L^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu)$. Then, there exists a constant $\mathcal{C}_k = C_{d,\gamma,\beta,k}>0$ such that for any $t\in\mathbb{R}_+$ it holds*

$$\|f(t,\cdot,\cdot)\|_{L^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu)} \leq \mathcal{C}_k \|f_{\mathrm{in}}\|_{L^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu)}.$$

This strategy of using the conservation of weighted norms has also been used in a companion paper when F had a sub-exponential form. There, any value of k was authorized, and this was implicitly a consequence of the fact that such a F had finite weighted norms $L^2(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}\mu)$ for any $k \in \mathbb{R}_+$. In the present case of this chapter, one has to be really careful with the order of the weighted norms at stage. Note that we may not expect propagation of higher moments than those that F has, and this justifies the range of values of k.

We thus choose to adopt the following interpolation strategy. Note that for any function $h \in L^2(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}\mu)$, one has immediately

$$\|h\|_{L^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu)} = \left\|F^{-1}h\right\|_{L^2(F\langle v\rangle^k \mathrm{d}x\,\mathrm{d}v)}.$$

As such, it is equivalent to control the semi-group $e^{(\mathsf{L}-\mathsf{T})t}$ in $L^2(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}\mu)$ and $F^{-1}e^{(\mathsf{L}-\mathsf{T})t}$ in $L^2(F\,\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}v)$. Since $L^2(\langle v\rangle^k F\mathrm{d}x\,\mathrm{d}v)$ is an interpolation space between $L^1(F\,\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}v)$ and $L^\infty(\mathrm{d}x\,\mathrm{d}v)$ (see [201]), if we prove that $F^{-1}e^{(\mathsf{L}-\mathsf{T})t}$ is bounded onto $L^\infty(\mathrm{d}x\,\mathrm{d}v)$ and onto $L^1(F\,\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}v)$, it will be automatically bounded onto $L^2(F\,\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}v)$, which is exactly the result of Proposition 5.9.

The rest of this section thus goes as follows. We will separately show boundedness of $F^{-1}e^{(\mathsf{L}-\mathsf{T})t}$ onto $L^\infty(\mathrm{d}x\,\mathrm{d}v)$ and onto $L^1(F\,\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}v)$. The first point is relatively immediate whereas the second point requires a Lyapunov and splitting strategy, as previously used in [130, 132] and Chapter 4.

5.3.1 The boundedness in $L^\infty(\mathrm{d}x\,\mathrm{d}v)$

Lemma 5.10. *Let* L *be as in case* (a), (b) *or* (c). *Then, in all these cases,*

$$\left\|F^{-1}e^{t(\mathsf{L}-\mathsf{T})}\right\|_{L^\infty(\mathrm{d}x\,\mathrm{d}v)\to L^\infty(\mathrm{d}x\,\mathrm{d}v)} \le 1.$$

Proof. This is a consequence of the maximum principle. □

5.3.2 The boundedness in $L^1(F\,\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}v)$

Remark that bounding the operator $F^{-1}e^{t(\mathsf{L}-\mathsf{T})}$ in $L^1(F\,\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}v)$ is equivalent to bounding $e^{t(\mathsf{L}-\mathsf{T})}$ in $L^1(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}v)$. To obtain such a bound, we first write a Lyapunov type estimate.

Lemma 5.11. *Let* L *be as in case* (a), (b) *or* (c). *Then, in all these cases, for any* $k \in (0, \gamma-\beta)$, *there exists* $(a, b, R) \in \mathbb{R}\times\mathbb{R}_+^2$ *such that for any* $f \in L^1(\langle v\rangle^k)$ *the following inequality holds*

$$\iint_{\mathbb{R}^{2d}} \tfrac{f}{|f|}\,\mathsf{L}f\,\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}v \le \iint_{\mathbb{R}^{2d}} \left(a\,\mathbb{1}_{B_R} - b\,\langle v\rangle^\beta\right)|f|\,\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}v.$$

Proof. First assume that $f \ge 0$. Then one may write,

$$\iint_{\mathbb{R}^{2d}} \mathsf{L}f\,\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}v = \iint_{\mathbb{R}^{2d}} \mathsf{L}f\,F\,\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}\mu = \iint_{\mathbb{R}^{2d}} F^{-1}\mathsf{L}^*(F\,\langle v\rangle^k)\,f\,\mathrm{d}x\,\mathrm{d}v.$$

In case (a),

$$\begin{aligned} F^{-1}\mathsf{L}^*(F\langle\cdot\rangle^k)(v) &= F^{-1}\mathsf{L}(F\langle\cdot\rangle^k)(v) \\ &= \nabla_v\cdot\Big(\langle v\rangle^{-d-\gamma}\nabla_v\Big(\langle v\rangle^k\Big)\Big) \\ &= k\nabla_v\cdot\Big(\langle v\rangle^{-d-\gamma+k-2}v\Big)F^{-1} \\ &= kd\langle v\rangle^{k-2} + k|v|^2(-d-\gamma+k-2)\langle v\rangle^{k-4} \\ &= k(k-\gamma-2)\langle v\rangle^{k-2} + k(d+\gamma-k+2)\langle v\rangle^{k-4} \end{aligned}$$

which indeed implies the result for $k<\gamma-\beta$ since in this case $\beta=-2$.

In case (b), using assumption (H1), one obtains

$$\begin{aligned} F^{-1}\mathsf{L}^*(F\langle\cdot\rangle^k)(v) &= \int_{\mathbb{R}^d}\mathrm{b}(v',v)\Big(\langle v'\rangle^k F(v') - \langle v\rangle^k F(v')\Big)\,\mathrm{d}v' \\ &= \left(\int_{\mathbb{R}^d}\mathrm{b}(v',v)\frac{\langle v'\rangle^k}{\langle v\rangle^k}F(v')\,\mathrm{d}v' - \nu(v)\right)\langle v\rangle^k. \end{aligned}$$

By hypothesis (H4), it yields

$$F^{-1}\mathsf{L}^*(F\langle\cdot\rangle^k)(v) \le \left(\frac{\mathcal{C}_{\mathrm{b}}}{\langle v\rangle^k} - 1\right)\langle v\rangle^\beta.$$

We conclude that inequality (5.11) holds for any $k<\gamma-\beta$.

In case (c),

$$\begin{aligned} F^{-1}\mathsf{L}^*(F\langle\cdot\rangle^k)(v) &= \Delta_v^{\frac{\mathfrak{a}}{2}}(\langle\cdot\rangle^k) - E(v)\cdot\nabla_v\Big(\langle v\rangle^k\Big) \\ &= \Big(\langle v\rangle^{-k}\Delta_v^{\frac{\mathfrak{a}}{2}}(\langle\cdot\rangle^k) - k(v\cdot E(v))\langle v\rangle^{-2}\Big)\langle v\rangle^k \\ &\le \Big(\langle v\rangle^{-k}\Delta_v^{\frac{\mathfrak{a}}{2}}(\langle\cdot\rangle^k) - C_{F,\mathfrak{a}}k\langle v\rangle^\beta\Big)\langle v\rangle^k, \end{aligned}$$

where we used Formula (5.4) that gives $v\cdot E \ge C_{F,\mathfrak{a}}\langle v\rangle^{\beta+2}$. Since $k<\mathfrak{a}=\gamma-\beta$, we control $\langle v\rangle^{-k}\Delta_v^{\frac{\mathfrak{a}}{2}}(\langle\cdot\rangle^k)$ using a classical estimate for the fractional Laplacian of weights (see e.g. [32, 31, 132]),

$$\langle v\rangle^{-k}\Delta_v^{\frac{\mathfrak{a}}{2}}(\langle\cdot\rangle^k) \lesssim \langle v\rangle^{-\mathfrak{a}}.$$

If now we remove the assumption that $f\ge 0$, the result still follows by using Kato's inequality (see e.g. [48] for the Laplacian and [69] for the fractional Laplacian) for the operator L since it implies

$$\iint_{\mathbb{R}^{2d}}\tfrac{f}{|f|}\mathsf{L}f\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v \le \iint_{\mathbb{R}^{2d}}\mathsf{L}(|f|)\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v,$$

and we can use the estimate by replacing f by $|f|$. □

5.3.3 A factorization method

We employ a "shrinking" strategy, as in [111, 130, 167]. We write $\mathsf{L}-\mathsf{T}$ as a dissipative part C and a bounded part B such that $\mathsf{L}-\mathsf{T}=\mathsf{B}+\mathsf{C}$.

Lemma 5.12. *With the notation of Lemma 5.11, let* $(k,k_2)\in(0,\gamma)\times(0,\gamma-\beta)^2$ *such that* $k_2>k-\beta$, $a=\max(a_k,a_{k_2})$, $R=\max\left\{R_k,R_{k_2}\right\}$, $\mathsf{C}=a\,\mathbb{1}_{B_R}$ *and* $\mathsf{B}=\mathsf{L}-\mathsf{T}-\mathsf{C}$. *For any* $t\in\mathbb{R}_+$, *we have:*

(i) $\|\mathsf{C}\|_{L^1(\mathrm{d}x\,\mathrm{d}\mu)\to L^1\left(\langle v\rangle^{k_2}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\le a\,\langle R\rangle^{k_2}$,

(ii) $\|e^{t\mathsf{B}}\|_{L^1\left(\langle v\rangle^{k}\,\mathrm{d}x\,\mathrm{d}\mu\right)\to L^1\left(\langle v\rangle^{k}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\le 1$,

(iii) $\|e^{t\mathsf{B}}\|_{L^1\left(\langle v\rangle^{k_2}\,\mathrm{d}x\,\mathrm{d}\mu\right)\to L^1\left(\langle v\rangle^{k}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\le C\,(1+t)^{\frac{k_2-k}{\beta}}$ *for some* $C>0$.

Proof. Property (i) is a consequence of the definition of C. Property (ii) follows from Lemma 5.11. Indeed, for any $g\in L^1(\langle v\rangle^k)$,

$$\iint_{\mathbb{R}^{2d}}\mathsf{B}g\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v\le\iint_{\mathbb{R}^{2d}}\left(a_k\,\mathbb{1}_{B_{R_k}}-a\,\mathbb{1}_{B_R}-b_k\,\langle v\rangle^{\beta}\right)|g|\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v$$
$$\le -\,b_k\,\|g\|_{L^1\left(\langle v\rangle^{k-\beta}\,\mathrm{d}x\,\mathrm{d}v\right)}\,.$$

To prove (iii), define $g=e^{t\mathsf{B}}f_{\mathrm{in}}$. By Hölder's inequality and the above contraction property, we get

$$\|g\|_{L^1\left(\langle v\rangle^{k}\,\mathrm{d}v\,\mathrm{d}x\right)}\le\|g\|^{\frac{k_2-k}{k_2-k-\beta}}_{L^1\left(\langle v\rangle^{k-\beta}\,\mathrm{d}x\,\mathrm{d}v\right)}\left\|f^{\mathrm{in}}\right\|^{\frac{|\beta|}{k_2-k-\beta}}_{L^1\left(\langle v\rangle^{k_2}\,\mathrm{d}x\mathrm{d}v\right)}\,.$$

As a consequence,

$$\iint_{\mathbb{R}^{2d}}\mathsf{B}f\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v\le -b_k\,\left(\|g\|_{L^1\left(\langle v\rangle^{k}\,\mathrm{d}v\,\mathrm{d}x\right)}\right)^{1+\frac{|\beta|}{k_2-k}}\left\|f^{\mathrm{in}}\right\|^{-\frac{|\beta|}{k_2-k}}_{L^1\left(\langle v\rangle^{k_2}\,\mathrm{d}x\mathrm{d}v\right)}\,,$$

so that by Grönwall's Lemma, we obtain

$$\|f\|_{L^1(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}v)}\le\left(\left\|f^{\mathrm{in}}\right\|^{-\frac{|\beta|}{k_2-k}}_{L^1(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}v)}+\frac{k_2-k}{b_k|\beta|}t\left\|f^{\mathrm{in}}\right\|^{-\frac{|\beta|}{k_2-k}}_{L^1(\langle v\rangle^{k_2}\mathrm{d}x\,\mathrm{d}v)}\right)^{-\frac{k_2-k}{|\beta|}}$$
$$\le\frac{1}{\left(1+\frac{b_k|\beta|}{k_2-k}t\right)^{\frac{k_2-k}{|\beta|}}}\left\|f^{\mathrm{in}}\right\|_{L^1(\langle v\rangle^{k_2}\mathrm{d}x\,\mathrm{d}v)}\,,$$

which implies (iii). □

Lemma 5.13. *Let* $k\in(0,\gamma)$ *and* f *be a solution of (5.1) with initial condition* $f^{\mathrm{in}}\in L^1(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v)$. *Then, there exists a constant* $\mathcal{C}_k=C_{d,\gamma,\beta,k}>0$ *such that*

$$\|f\|_{L^1(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}v)}\le\mathcal{C}_k\left\|f^{\mathrm{in}}\right\|_{L^1(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}v)}\,.$$

Proof. Defining the convolution of two operators by $\mathsf{U} \star \mathsf{V} = \int_0^t \mathsf{U}(t-s)\mathsf{V}(s)\,\mathrm{d}s$, we can write the following Duhamel's formula

$$e^{t(\mathsf{L}-\mathsf{T})} = e^{t\mathsf{B}} + e^{t\mathsf{B}} \star \mathsf{C}\, e^{t(\mathsf{L}-\mathsf{T})}.$$

Combining the formulas from Lemma 5.12 and the fact that

$$\|e^{t(\mathsf{L}-\mathsf{T})}\|_{L^1(\mathrm{d}x\,\mathrm{d}v)\to L^1(\mathrm{d}x\,\mathrm{d}v)} \leq 1,$$

we get

$$\|e^{t\mathsf{L}}\|_{L^1(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}v)\to L^1(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}v)} \leq 1 + a\,\langle R\rangle^{k_2} \int_0^t \frac{\mathrm{d}s}{(1+cs)^{\frac{k_2-k}{|\beta|}}},$$

which is bounded uniformly in time since $k_2 - k > -\beta = |\beta|$. □

5.4 Proof of Theorems 5.4 and 5.5

To prove the result, we come back to the space variable x. For this, integrate the result of Proposition 5.6 with respect to ξ, and use Plancherel's formula, to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}(f) \leq -\frac{\delta}{2}\left(\int_{\mathbb{R}^d} \frac{|\xi|^{\alpha'}}{1+|\xi|^{\alpha'}} \left\|\mathsf{\Pi}\widehat{f}\right\|^2 \mathrm{d}\xi + \|(1-\mathsf{\Pi})f\|^2_{L^2(\langle v\rangle^\beta \mathrm{d}x\,\mathrm{d}\mu)}\right).$$

To control the macroscopic part, we shall control

$$\int_{\mathbb{R}^d} \frac{|\xi|^{\alpha'}}{1+|\xi|^{\alpha'}} \left\|\mathsf{\Pi}\widehat{f}\right\|^2 \mathrm{d}\xi,$$

in terms of $\|\mathsf{\Pi} f\|^2$. This is done in the following lemma.

Lemma 5.14. *For any $f \in L^2(\mathrm{d}x\,\mathrm{d}\mu) \cap L^1(\mathrm{d}x\,\mathrm{d}v)$,*

$$\|\mathsf{\Pi} f\|^2_{L^2(\mathrm{d}x\,\mathrm{d}\mu)} \lesssim \int_{\mathbb{R}^d} \frac{|\xi|^{\alpha'}}{1+|\xi|^{\alpha'}} \left\|\mathsf{\Pi}\widehat{f}\right\|^2 \mathrm{d}\xi + \|f\|^{\frac{2\alpha'}{d+\alpha'}}_{L^1(\mathrm{d}x\mathrm{d}v)} \left(\int_{\mathbb{R}^d} \frac{|\xi|^{\alpha'}}{1+|\xi|^{\alpha'}} \left\|\mathsf{\Pi}\widehat{f}\right\|^2 \mathrm{d}\xi\right)^{\frac{d}{d+\alpha'}}.$$

Proof. Define u_f through

$$\forall \xi \in \mathbb{R}^d, \qquad \left(1+|\xi|^{\alpha'}\right)\widehat{u_f} = \widehat{\rho_f}.$$

From this definition, one sees that

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{|\xi|^{\alpha'}}{1+|\xi|^{\alpha'}} \left\|\mathsf{\Pi}\widehat{f}\right\|^2 \mathrm{d}\xi &= \int_{\mathbb{R}^d} |\xi|^{\alpha'}\left(1+|\xi|^{\alpha'}\right)\widehat{u_f}\,\overline{\widehat{u_f}}\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} ||\xi|^{\frac{\alpha'}{2}}\widehat{u_f}|^2\,\mathrm{d}\xi + \int_{\mathbb{R}^d} ||\xi|^{\alpha'}\widehat{u_f}|^2\,\mathrm{d}\xi,\end{aligned}$$

Moreover, after taking the modulus and squaring, it also gives

$$\begin{aligned}\|\Pi f\|^2 &= \|\widehat{u_f}\|^2 + 2\left\||\xi|^{\frac{\alpha'}{2}}\widehat{u_f}\right\|^2 + \left\||\xi|^{\alpha'}\widehat{u_f}\right\|^2\\ &\le \|\widehat{u_f}\|^2 + 2\int_{\mathbb{R}^d}\frac{|\xi|^{\alpha'}}{1+|\xi|^{\alpha'}}\left\|\Pi\widehat{f}\right\|^2 \mathrm{d}\xi,\end{aligned}$$

Now using the fractional Nash's inequality (5.7) on u_f, we get

$$\begin{aligned}\|\widehat{u_f}\|^2_{L^2(\mathrm{d}x)} = \|u_f\|^2_{L^2(\mathrm{d}x)} &\le C\,\|\rho\|^{\frac{2\alpha'}{d+\alpha'}}_{L^1(\mathrm{d}x)}\left\||\xi|^{\frac{\alpha'}{2}}\widehat{u_f}\right\|^{\frac{2d}{d+\alpha'}}_{L^2}\\ &\le \mathcal{C}_{\mathrm{Nash}}\,\|u_f\|^{\frac{2\alpha'}{d+\alpha'}}_{L^1(\mathrm{d}x)}\left(\int_{\mathbb{R}^d}\mu_2(\xi)\left\|\Pi\widehat{f}\right\|^2\mathrm{d}\xi\right)^{\frac{2d}{d+\alpha'}}.\end{aligned}$$

The identity $\|u_f\|_{L^1(\mathrm{d}x)} = \|\rho_f\|_{L^1(\mathrm{d}x)} = \|f\|_{L^1(\mathrm{d}x\,\mathrm{d}v)}$. Combining the identities leads to

$$\begin{aligned}\|\Pi f\|^2 \le\ & \mathcal{C}_{\mathrm{Nash}}\,\|f_{\mathrm{in}}\|^{\frac{2\alpha'}{d+\alpha'}}_{L^1(\mathrm{d}x\,\mathrm{d}v)}\left(\int_{\mathbb{R}^d}\frac{|\xi|^{\alpha'}}{1+|\xi|^{\alpha'}}\left\|\Pi\widehat{f}\right\|^2\mathrm{d}\xi\right)^{\frac{d}{d+\alpha'}}\\ &+ 2\int_{\mathbb{R}^d}\frac{|\xi|^{\alpha'}}{1+|\xi|^{\alpha'}}\left\|\Pi\widehat{f}\right\|^2\mathrm{d}\xi.\end{aligned}$$

This defines Φ such that $\Phi(\|\Pi f\|^2) \le \int_{\mathbb{R}^d}\frac{|\xi|^{\alpha'}}{1+|\xi|^{\alpha'}}\left\|\Pi\widehat{f}\right\|^2\mathrm{d}\xi$, with $\Phi^{-1}(y) = y + \left(\frac{y}{\mathsf{c}}\right)^{\frac{d}{d+\alpha'}}$ for c given through above. □

We now need to control the microscopic part. This is given by the following lemma.

Lemma 5.15. *Assume $\beta > 0$. Then for any $f \in L^2(\mathrm{d}x\,\mathrm{d}\mu)$*

$$\|(1-\Pi)f\|^2_{L^2(\langle v\rangle^\beta\mathrm{d}x\,\mathrm{d}\mu)} \ge \|(1-\Pi)f\|^2.$$

Assume $\beta \le 0$. Then for any $f \in L^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)\cap L^1(\mathrm{d}x\,\mathrm{d}v)$,

$$\|(1-\Pi)f\|^2_{L^2(\langle v\rangle^\beta\mathrm{d}x\,\mathrm{d}\mu)} \ge \Big((1+\Theta_k)\,\|f\|_{L^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu)}\Big)^{\frac{2\beta}{k}}\Big(\|(1-\Pi)f\|^2\Big)^{1-\frac{\beta}{k}}.$$

Proof. The case $\beta > 0$ is immediate since then $\langle v\rangle^\beta \ge 1$ for all v. For the case $\beta < 0$, use Hölder's inequality to get

$$\|(1-\Pi)f\| \le \|(1-\Pi)f\|^{\frac{k}{k-\beta}}_{L^2(\langle v\rangle^\beta\mathrm{d}x\,\mathrm{d}\mu)}\,\|(1-\Pi)f\|^{\frac{-\beta}{k-\beta}}_{L^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu)}, \tag{5.14}$$

and note that

$$\begin{aligned}\|(1-\Pi)f\|_{L^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu)} &\le \|f\|_{L^2(\langle v\rangle^k\mathrm{d}\mu)} + \Theta_k\,\|\rho\|_{L^2(\mathrm{d}x)}\\ &\le (1+\Theta_k)\,\|f\|_{L^2(\langle v\rangle^k\mathrm{d}\mu\mathrm{d}x)},\end{aligned}$$

□

The conclusion of the proof of Theorems 5.4 and 5.5 is now as in [44, 45] and Chapter 4. For legibility, we shall split cases according to the sign of β.

If β is positive, then using the estimates of Lemma 5.14 and Lemma 5.15, we obtain that

$$\|(1-\Pi)f\|^2_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)} + \int_{\mathbb{R}^d} \frac{|\xi|^{\alpha'}}{1+|\xi|^{\alpha'}} \left\|\Pi\widehat{f}\right\|^2 \mathrm{d}\xi \geq \|(1-\Pi)f\|^2 + \Phi\Big(\|\Pi f\|^2\Big)\,.$$

Since

$$\|(1-\Pi)f\|^2 \leq z_0 \quad \text{and} \quad \|\Pi f\|^2 \leq z_0 \quad \text{where} \quad z_0 := \frac{1+\delta}{1-\delta}\,\|f_{\rm in}\|^2.$$

and from

$$\Phi^{-1}(y) = y + \left(\frac{y}{\mathsf{c}}\right)^{\frac{d}{d+\alpha'}} \leq \left(C_1^{-1}\,y\right)^{\frac{d}{d+\alpha'}}$$
$$\text{with} \quad C_1 := \left(\Phi(z_0)^{\frac{2}{d+\alpha'}} + \mathsf{c}^{-\frac{d}{d+\alpha'}}\right)^{-\frac{d+\alpha'}{d}}\,,$$

as long as $y \leq \Phi(z_0)$, we obtain

$$\Phi\Big(\|\Pi f\|^2\Big) \geq C_1\,\|\Pi f\|^{2\,\frac{d+\alpha'}{d}},$$

since $\|\Pi f\|^2 \leq z_0$. As a consequence,

$$\begin{aligned}\|(1-\Pi)f\|^2_{L^2(\langle v\rangle^\beta \mathrm{d}\mu)} &+ \int_{\mathbb{R}^d} \frac{|\xi|^{\alpha'}}{1+|\xi|^{\alpha'}} \left\|\Pi\widehat{f}\right\|^2 \mathrm{d}\xi \\ &\geq \|(1-\Pi)f\|^2 + C_1\,\|\Pi f\|^{2\,\frac{d+\alpha'}{d}}. \\ &\geq \min\{1, C_1\}\,\|f\|^{2(1+\frac{\alpha'}{d})}\,.\end{aligned}$$

Collecting terms, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}(f) \leq -\,C\,\mathsf{H}(f)^{1+\alpha'/d},$$

using the norm equivalence between $\|f\|$ and $\mathsf{H}(f)$. Then the proof of the theorem follows from a Grönwall estimate.

If β is nonpositive, the proof is actually very similar, except that one has to use the conservation of weighted norms given by Proposition 5.9 for $k \leq \gamma - \beta$ to control $\|f\|_{L^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu)}$. We obtain in this case

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}(f) \leq -\,C\,\mathsf{H}(f)^{1+1/\zeta},$$

with $\zeta := \max\Big(\frac{\alpha'}{d}, \frac{|\beta|}{k}\Big)$, and then the proof of the theorems follows from a Grönwall estimate. The numerology found in Theorems 5.4 and 5.5 follows from the fact that

$$\alpha' = 2 + \frac{\min(\gamma+\beta-2, 0)}{|1-\beta|} = 2,$$

if $\gamma + \beta > 2$, and that if $\gamma + \beta < 2$, then necessarily $\beta < 1$ since $\gamma > \beta$ and thus

$$\alpha' = 2 + \frac{\min(\gamma+\beta-2, 0)}{|1-\beta|} = 2 + \frac{\gamma+\beta-2}{1-\beta} = \frac{\gamma-\beta}{1-\beta} = \alpha.$$

5.5 Quantitative estimates of μ_{L} and λ_{L}

In this section, we show that the three types of operators we consider in the chapter satisfy the hypothesis on μ_{L} and λ_{L} needed in 5.6, that is

$$\mu_{\mathsf{L}}(\xi) \lesssim |\xi|^{\min\left(1, 1+\frac{\gamma+\beta-2}{2|1-\beta|}\right)} \mathbb{1}_{|\xi|\le 1} + \mathbb{1}_{|\xi|\ge 1}, \qquad \lambda_{\mathsf{L}}(\xi) \lesssim 1.$$

For readability, let us recall that

$$\mu_{\mathsf{L}}(\xi) := \left\| \mathsf{L}^* \left((v\cdot\xi)\varphi(\xi,\cdot) F \right) \right\|_{L^2(\langle v\rangle^{-\beta} \mathrm{d}\mu)}, \qquad \lambda_{\mathsf{L}}(\xi) := \left\| \mathsf{L}^* \left(\psi(\xi,\cdot) F \right) \right\|_{L^2(\langle v\rangle^{-\beta} \mathrm{d}\mu)}.$$

5.5.1 Fokker-Planck operators

Lemma 5.16. *Consider $\mathsf{L} = \mathsf{L}_1$, a Fokker-Planck operator. Then*

$$\mu_{\mathsf{L}} \lesssim |\xi|^{\min\left(1, \frac{\gamma+2}{6}\right)} \mathbb{1}_{|\xi|\le 1} + |\xi|^{-1} \mathbb{1}_{|\xi|\ge 1}, \qquad \lambda_{\mathsf{L}} \lesssim 1.$$

Proof. Note that in this case, L is self-adjoint. Start with estimating μ_{L}.

$$\begin{aligned} F^{-1}\mathsf{L}\left((v\cdot\xi)\varphi F\right) &= \nabla_v \cdot \left(F \nabla_v \left((v\cdot\xi)\varphi\right)\right) \\ &= \Delta_v((v\cdot\xi)\varphi)) - (d+\gamma)\frac{v}{\langle v\rangle^2} \cdot \nabla_v((v\cdot\xi)\varphi). \end{aligned}$$

We can then compute

$$\begin{aligned} \nabla_v((v\cdot\xi)\varphi) &= \varphi\,\xi + (v\cdot\xi)\nabla_v\varphi \\ \Delta_v((v\cdot\xi)\varphi) &= 2\,\xi\cdot\nabla_v\varphi + (v\cdot\xi)\Delta_v\varphi. \end{aligned}$$

We end up with

$$F^{-1}\mathsf{L}\left((v\cdot\xi)\varphi\right) = 2\xi\cdot\nabla_v\varphi + (v\cdot\xi)\left(\Delta_v\varphi - \frac{(d+\gamma)}{\langle v\rangle^2}(\varphi + v\cdot\nabla_v\varphi)\right).$$

Recalling that $\varphi = \frac{\langle v\rangle^2}{1+\langle v\rangle^6|\xi|^2}$, we may for legibility write $A = 1 + \langle v\rangle^6 |\xi|^2$ so that $\varphi = \langle v\rangle^2 A^{-1}$. This yields

$$\nabla_v\varphi = \left(2A^{-1} - 6\langle v\rangle^6 |\xi|^2 A^{-2}\right) v = 2\left(1 - 2\langle v\rangle^6 |\xi|^2\right)\frac{v}{A^2}.$$

As a consequence,

$$\begin{aligned} \xi\cdot\nabla_v\varphi &= 2\left(1 - 2\langle v\rangle^6 |\xi|^2\right)\frac{v\cdot\xi}{A^2} \\ v\cdot\nabla_v\varphi &= 2\left(1 - 2\langle v\rangle^6 |\xi|^2\right)\frac{|v|^2}{A^2}. \end{aligned}$$

From this we can readily estimate, since $\langle v\rangle^6 |\xi|^2 \le A$,

$$|\xi\cdot\nabla_v\varphi| \lesssim |v\cdot\xi| A^{-1}, \qquad |v\cdot\nabla_v\varphi| \lesssim \langle v\rangle^2 A^{-1}.$$

The last part to estimate is

$$\begin{aligned}\Delta_v \varphi &= 2\left(1 - 2\langle v\rangle^6 |\xi|^2\right) \nabla_v \cdot \left(\frac{v}{A^2}\right) + 2\nabla_v \cdot \left(1 - 2\langle v\rangle^6 |\xi|^2\right)\frac{v}{A^2}, \\ &= \frac{2}{A^2}\left(1 - 2\langle v\rangle^6 |\xi|^2\right)\left(d + 12|v|^2 \langle v\rangle^4 |\xi|^2 A^{-1}\right) - 24|v|^2 \langle v\rangle^4 |\xi|^2 A^{-2}\end{aligned}$$

that gives

$$|\Delta\varphi| \lesssim A^{-1}.$$

Combining previous estimates, we thus end up with

$$\left|F^{-1}\mathsf{L}\left((v\cdot\xi)\varphi\right)\right| \lesssim |v\cdot\xi| A^{-1}.$$

This allows to estimate $\mu_{\mathsf{L}}(\xi)$ as follows,

$$\|\mathsf{L}\left((v\cdot\xi)\varphi(\xi,\cdot)F\right)\|_{L^2(\langle v\rangle^2 \mathrm{d}\mu)} \lesssim \left(\int_{\mathbb{R}^d} \frac{|v\cdot\xi|^2}{\left(1+\langle v\rangle^6 |\xi|^2\right)^2} \frac{\langle v\rangle^2 \,\mathrm{d}v}{\langle v\rangle^{d+\gamma}}\right)^{\frac12}.$$

The conclusion then comes from following exactly the same steps as in the proof of Lemma 5.7, that gives

$$\mu_{\mathsf{L}} \lesssim |\xi|^{\min\left(1,\frac{2+\gamma}{6}\right)} \mathbb{1}_{|\xi|\le 1} + |\xi|^{-1}\mathbb{1}_{|\xi|\ge 1}.$$

We now estimate of λ_{L}. Recalling that $\varphi_0 = (1+\langle v\rangle^2 |\xi|^2)^{-1}$, one has

$$F^{-1}\mathsf{L}\left(\varphi_0 F\right) = \Delta_v\varphi_0 - (d+\gamma)\frac{v}{\langle v\rangle^2}\cdot\nabla_v\varphi_0.$$

But

$$\frac{v}{\langle v\rangle^2}\cdot\nabla_v\varphi_0 = -2\frac{|v|^2}{\langle v\rangle^2}|\xi|^2\varphi_0^2, \qquad \Delta_v\varphi = \left(8|v|^2|\xi|^2\varphi_0 - 2d\right)|\xi|^2\varphi_0^2.$$

that gives

$$\left|F^{-1}\mathsf{L}\left(\varphi_0 F\right)\right| \lesssim |\xi|^2\varphi_0^2,$$

and thus

$$\left|F^{-1}\mathsf{L}\left(\psi F\right)\right| \lesssim \frac{|\xi|^2}{\|\varphi_0 F\|}\varphi_0^2.$$

Since $(1+|\xi|^2)^{-1} \lesssim \|\varphi_0 F\|$, we now estimate

$$\begin{aligned}\lambda_{\mathsf{L}} &\lesssim (1+|\xi|^2)|\xi|^2\left(\int_{\mathbb{R}^d}\frac{1}{\left(1+\langle v\rangle^2|\xi|^2\right)^4}\frac{\langle v\rangle^2\,\mathrm{d}v}{\langle v\rangle^{d+\gamma}},\right)^{\frac12} \\ &\lesssim (1+|\xi|^2)|\xi|^2\left(|\xi|^{\min\left(0,\frac{\gamma-2}{2}\right)}\mathbb{1}_{|\xi|\le1} + (1+|\xi|^2)^{-2}\mathbb{1}_{|\xi|\ge1}\right), \\ &\lesssim |\xi|^{2+\min\left(0,\frac{\gamma-2}{2}\right)}\mathbb{1}_{|\xi|\le1} + \mathbb{1}_{|\xi|\ge1}.\end{aligned}$$

following exactly the same steps as in the proof of Lemma 5.7. □

5.5.2 Scattering collision operators

Lemma 5.17. *Let $\mathsf{L} = \mathsf{L}_2$ and $\gamma > \beta$. Then*

$$\mu_{\mathsf{L}} \lesssim |\xi|^{\min\left(1, 1+\frac{\gamma+\beta-2}{2|1-\beta|}\right)} \mathbb{1}_{|\xi|<1} + |\xi|^{-1} \mathbb{1}_{|\xi|\geq 1}, \qquad \lambda_{\mathsf{L}} \lesssim 1.$$

Proof. To estimate μ_{L}, we write

$$\begin{aligned} F^{-1}\mathsf{L}^*\left((v\cdot\xi)\varphi F\right) &= \int_{\mathbb{R}^d} \mathrm{b}(v',v)\left((v'\cdot\xi)\varphi(v') - (v\cdot\xi)\varphi(v)\right) F(v')\,\mathrm{d}v' \\ &= \int_{\mathbb{R}^d} \mathrm{b}(v',v)(v'\cdot\xi)\varphi(v')\,F(v')\,\mathrm{d}v' - (v\cdot\xi)\varphi(v)\nu(v). \end{aligned}$$

Then we first remark that the inequality of Cauchy-Schwarz yields

$$\begin{aligned} &\int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} \mathrm{b}(v',v)(v'\cdot\xi)\varphi(v')F'\,\mathrm{d}v'\right|^2 \langle v\rangle^{-\beta} F\,\mathrm{d}v \\ &\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |(v'\cdot\xi)\varphi(v')|^2 \langle v\rangle^{\beta} F'\,\mathrm{d}v'\right)\left(\int_{\mathbb{R}^d} \frac{\mathrm{b}(v',v)^2}{\nu(v')} F'\,\mathrm{d}v'\right) \langle v\rangle^{-\beta} F\,\mathrm{d}v \\ &\leq \mathcal{C}_{\mathrm{b}} \int_{\mathbb{R}^d} |\nu(v)(v\cdot\xi)\varphi(v)|^2 \langle v\rangle^{-\beta} F\,\mathrm{d}v, \end{aligned}$$

where, by assumption (H3),

$$\mathcal{C}_{\mathrm{b}} := \iint_{\mathbb{R}^{2d}} \frac{\mathrm{b}(v',v)^2}{\nu(v')\nu(v)} F F'\,\mathrm{d}v\,\mathrm{d}v' < \infty.$$

Then, since

$$\int_{\mathbb{R}^d} |\nu(v)(v\cdot\xi)\varphi|^2 \langle v\rangle^{-\beta} F\,\mathrm{d}v \leq C \int_{\mathbb{R}^d} \frac{|v\cdot\xi|^2}{\left(1+\langle v\rangle^{2|1-\beta|}|\xi|^2\right)^2} \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\beta}},$$

the result follows again by adapting the proof of Lemma 5.7. The estimate for $\lambda_{\mathsf{L}}(\xi)$ comes fairly easily from

$$\begin{aligned} F^{-1}\mathsf{L}^*\left(\varphi_0 F\right) &= \int_{\mathbb{R}^d} \mathrm{b}(v',v)\left(\varphi_0(v') - \varphi_0(v)\right) F(v')\,\mathrm{d}v' \\ &= \int_{\mathbb{R}^d} \mathrm{b}(v',v)\,\varphi_0(v')\,F(v')\,\mathrm{d}v' - \nu(v)\varphi_0(v). \end{aligned}$$

Again, the inequality of Cauchy-Schwarz yields

$$\begin{aligned} &\int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} \mathrm{b}(v',v)\varphi_0(v')F'\,\mathrm{d}v'\right|^2 \langle v\rangle^{-\beta} F\,\mathrm{d}v \\ &\qquad \leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\varphi_0(v')|^2 \langle v'\rangle^{\beta} F'\,\mathrm{d}v'\right)\left(\int_{\mathbb{R}^d} \frac{\mathrm{b}(v',v)^2}{\nu(v')} F'\,\mathrm{d}v'\right) \langle v\rangle^{-\beta} F\,\mathrm{d}v \\ &\qquad \leq \mathcal{C}_{\mathrm{b}} \int_{\mathbb{R}^d} |\nu(v)\varphi_0(v)|^2 \langle v\rangle^{-\beta} F\,\mathrm{d}v. \end{aligned}$$

Then, since

$$\int_{\mathbb{R}^d} |\nu(v)\varphi_0(v)|^2 \langle v\rangle^{-\beta} F \,\mathrm{d}v \le C \int_{\mathbb{R}^d} \frac{1}{\left(1+\langle v\rangle^2 |\xi|^2\right)^2} \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma-\beta}},$$

and $\|\varphi_0 F\|^{-1} \le \langle \xi\rangle^2$ we deduce

$$\lambda_{\mathsf{L}} \lesssim \langle \xi\rangle^2 \left(\int_{\mathbb{R}^d} \frac{1}{\left(1+\langle v\rangle^2 |\xi|^2\right)^2} \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma-\beta}}\right)^{\frac{1}{2}} \lesssim 1,$$

since $\gamma > \beta$. □

5.5.3 Fractional Fokker-Planck operators

In this section, we set $\mathsf{L} = \mathsf{L}_3$. Since computations are more involved, we split the estimates of μ_{L} and λ_{L}. We recall that since $\beta = \gamma - \mathfrak{a}$ with $\mathfrak{a} \in (0,2)$ and $\gamma + \beta > 0$, this implies $\beta > -1$.

Proposition 5.18. *For any $\gamma > |\beta|$ it holds*

$$|\Delta^{\frac{\mathfrak{a}}{2}}((v\cdot\xi)\varphi)| \lesssim |\xi|^{\min\left(1, 1+\frac{\gamma+\beta-2}{2|1-\beta|}\right)} \mathbb{1}_{|\xi|\le 1} + |\xi|^{-1}\mathbb{1}_{|\xi|\ge 1}, \tag{5.15}$$

which implies the same estimate on μ_{L}

$$\mu_{\mathsf{L}} \lesssim |\xi|^{\min\left(1, 1+\frac{\gamma+\beta-2}{2|1-\beta|}\right)} \mathbb{1}_{|\xi|\le 1} + \mathbb{1}_{|\xi|\ge 1}.$$

We separate the proof into two steps. We first need to split the fractional Laplacian in an appropriate way to get bounds depending on

$$m(v) = (v\cdot\xi)\varphi(\xi, v).$$

Since m has different behaviors depending on the value of ξ, we need two different ways of estimating the fractional Laplacian of this function to get the good dependency on ξ, which are proved in the following lemma.

Lemma 5.19. *For $0 < k < \mathfrak{a} \le 2$, the following relations hold*

$$|\Delta_v^{\frac{\mathfrak{a}}{2}} m| \le \frac{2^{\mathfrak{a}}\omega_d}{\langle v\rangle^{\mathfrak{a}}} \left(\frac{\langle v\rangle^2 \|\nabla_v^2 m\|_{L^\infty(B_v(R))}}{(2-\mathfrak{a})} + \frac{\langle v\rangle^k |m|_{C_v^{0,k}(B_v^c(R))}}{(\mathfrak{a}-k)}\right)$$

$$|\Delta_v^{\frac{\mathfrak{a}}{2}} m| \le \frac{2^{\mathfrak{a}}\omega_d}{\langle v\rangle^{\mathfrak{a}}} \left(\frac{\langle v\rangle^2 \|\nabla_v^2 m\|_{L^\infty(B_v(R))}}{(2-\mathfrak{a})} + \frac{\|m\|_{L^1(B_0(\langle v\rangle))}}{2^{-d}\omega_d \langle v\rangle^d} + \frac{\|m\|_{L^\infty(B_0^c(\langle v\rangle))}}{\mathfrak{a}}\right),$$

where we defined $R = \langle v\rangle/2$ and

$$|m|_{C_v^{0,k}(B_v^c(R))} := \sup_{|v-v'|>R} \left(\frac{|m(v)-m(v')|}{|v-v'|^k}\right).$$

Proof. We can write the fractional Laplacian as

$$\Delta_v^{\frac{\mathfrak{a}}{2}} m(v) = \iint_{|v-v'|<R} \frac{m(v') - m(v) - (v'-v)\cdot\nabla m(v)}{|v-v'|^{d+\mathfrak{a}}}\,\mathrm{d}v' + \iint_{|v-v'|\geq R} \frac{m(v')-m(v)}{|v-v'|^{d+\mathfrak{a}}}\,\mathrm{d}v',$$

with $R = \langle v\rangle/2$. The first integral is controlled by a second order Taylor approximation which yields

$$\begin{aligned}\left|\int_{|v-v'|\leq R} \frac{m(v') - m(v) - (v'-v)\cdot\nabla m(v)}{|v-v'|^{d+\mathfrak{a}}}\,\mathrm{d}v'\right| &\leq \int_{|z|\leq R} \frac{\|\nabla_v^2 m\|_{L^\infty(B_v(R))}}{|z|^{d+\mathfrak{a}-2}}\,\mathrm{d}z \\ &\leq \frac{2^{\mathfrak{a}-2}\omega_d \left\|\nabla_v^2 m\right\|_{L^\infty(B_v(R))}}{(2-\mathfrak{a})\langle v\rangle^{\mathfrak{a}-2}},\end{aligned}$$

where ∇^2 denotes the Hessian matrix of m. For the second integral, there are two ways of obtaining a bound. The first is more accurate for functions which grow to $+\infty$ when $|v|\to\infty$ and writes

$$\left|\int_{|v-v'|\geq R} \frac{m(v')-m(v)}{|v-v'|^{d+\mathfrak{a}}}\,\mathrm{d}v'\right| \leq \int_{|z|\geq R} \frac{|m|_{C_v^{0,k}}}{|z|^{d+\mathfrak{a}-k}}\,\mathrm{d}z \leq \frac{2^{\mathfrak{a}-k}\omega_d |m|_{C_v^{0,k}}}{(\mathfrak{a}-k)\langle v\rangle^{\mathfrak{a}-k}},$$

which proves the first assertion. When $m\in L^\infty$, we can also bound the part of the integral with $|v-v'|\geq R$ by splitting it once more to get

$$\begin{aligned}\left|\int_{\substack{|v-v'|\geq R\\ |v'|<\langle v\rangle}} \frac{m(v')}{|v-v'|^{d+\mathfrak{a}}}\mathrm{d}v'\right| &\leq \frac{2^{d+\mathfrak{a}}}{R^{d+\mathfrak{a}}}\int_{B_0(\langle v\rangle)} |m(v')|\,\mathrm{d}v' \leq \frac{2^{d+\mathfrak{a}}}{\langle v\rangle^{d+\mathfrak{a}}}\|m\|_{L^1(B_0(\langle v\rangle))} \\ \left|\int_{\substack{|v-v'|\geq R\\ |v'|\geq\langle v\rangle}} \frac{m(v')}{|v-v'|^{d+\mathfrak{a}}}\,\mathrm{d}v'\right| &\leq \|m\|_{L^\infty(B_0^c(\langle v\rangle))}\int_{|z|\geq R}\frac{\mathrm{d}z}{|z|^{d+\mathfrak{a}}} \leq \frac{2^{\mathfrak{a}}\omega_d\|m\|_{L^\infty(B_0^c(\langle v\rangle))}}{\mathfrak{a}\langle v\rangle^{\mathfrak{a}}} \\ \left|\int_{|v-v'|\geq R} \frac{m(v)}{|v-v'|^{d+\mathfrak{a}}}\,\mathrm{d}v'\right| &\leq |m(v)|\int_{|z|\geq R}\frac{\mathrm{d}z}{|z|^{d+\mathfrak{a}}} \leq \frac{2^{\mathfrak{a}}\omega_d\|m\|_{L^\infty(B_0^c(\langle v\rangle))}}{\mathfrak{a}\langle v\rangle^{\mathfrak{a}}}.\end{aligned}$$

□

We can now estimate each term in the right hand side the two formulas of Lemma 5.19.

Lemma 5.20. *For any $\xi\in\mathbb{R}^d$ and $\beta\geq -1$ it holds*

$$\left\|\nabla_v^2 m\right\|_{L^\infty(B_v(\langle v\rangle/2))} \lesssim |\xi|^{\frac{\alpha'}{2}}\langle v\rangle^{\frac{\alpha'(1-\beta)}{2}-2}\mathbb{1}_{|\xi|<1} + \mathbb{1}_{|\xi|\geq 1}\langle v\rangle^{-2}. \tag{5.16}$$

Moreover, for $\beta<1$ and $|\xi|<1$

$$|m|_{C_v^{0,k}(B_v^c(R))} \lesssim |\xi|^{\frac{\alpha'}{2}}, \tag{5.17}$$

and if $\beta\geq 1$ or $|\xi|\geq 1$

$$\|m\|_{L^\infty(B_0^c(\langle v\rangle))} \lesssim |\xi|^{\frac{\alpha'}{2}}\mathbb{1}_{|\xi|<1} + \mathbb{1}_{|\xi|\geq 1} \tag{5.18}$$

$$\|m\|_{L^1(B_0(\langle v\rangle))} \lesssim \langle v\rangle^d\left(|\xi|^{\frac{\alpha'}{2}}\mathbb{1}_{|\xi|<1} + \mathbb{1}_{|\xi|\geq 1}\right). \tag{5.19}$$

Proof of Lemma 5.20. We recall that we defined α' such that $\alpha' = \alpha$ if $\gamma + \beta \leq 2$ and $\alpha' = 2$ else. Now, remark that we can write

$$|\xi| \langle v \rangle \varphi = \frac{\langle v \rangle^{1-\beta} |\xi|}{1 + \langle v \rangle^{2|1-\beta|} |\xi|^2} = |\xi|^{\frac{\alpha'}{2}} \langle v \rangle^{\frac{\alpha'(1-\beta)}{2}} \frac{\langle v \rangle^{\frac{(1-\beta)(2-\alpha')}{2}} |\xi|^{\frac{2-\alpha'}{2}}}{1 + \langle v \rangle^{2|1-\beta|} |\xi|^2},$$

so that it holds

$$|\xi| \langle v \rangle \varphi \leq |\xi|^{\frac{\alpha'}{2}} \langle v \rangle^{\frac{\alpha'(1-\beta)}{2}} \mathbb{1}_{|\xi|<1} + \mathbb{1}_{|\xi|\geq 1}, \tag{5.20}$$

where we used Young's inequality or just the fact that $\langle v \rangle^{-1} \leq 1$ when $\beta \geq 1$.

- ***Inequality* (5.16)*: the Hessian bound***
  Computing the gradient of φ yields

$$\nabla_v \varphi = -\Big(\beta + (\beta + 2|1-\beta|) \langle v \rangle^{2|1-\beta|} |\xi|^2\Big) \langle v \rangle^{\beta - 2} \varphi^2 v,$$

from which we deduce that $|\nabla_v \varphi| \lesssim \langle v \rangle^{-1} \varphi$. Then to estimate the Hessian of φ, we write

$$\begin{aligned} \left|\nabla_v^2 (\varphi(v))\right| &= \left|\nabla_v \Big(\Big(\beta + (\beta + 2|1-\beta|) \langle v \rangle^{2|1-\beta|} |\xi|^2\Big) \langle v \rangle^{\beta-2} \varphi^2 v\Big)\right|, \\ &\lesssim \langle v \rangle^{-2} \varphi^2 + \langle v \rangle^{-1} |\nabla_v \varphi|, \end{aligned}$$

from which we deduce that $|\nabla_v^2 \varphi| \lesssim \langle v \rangle^{-2} \varphi$. Then, since $\nabla_v((v\cdot\xi)\varphi(v)) = \varphi(v)\xi + (v \cdot \xi)\nabla_v \varphi(v)$, it turns out that

$$\left|\nabla_v^2 ((v\cdot\xi)\varphi(v))\right| \lesssim |\nabla_v \varphi(v)||\xi| + |v\cdot\xi| \left|\nabla_v^2 (\varphi(v))\right| \lesssim |\xi| \langle v \rangle^{-1} \varphi.$$

Therefore, by (5.20), we obtain

$$\left|\nabla_v^2 m\right| \leq |\xi|^{\frac{\alpha'}{2}} \langle v \rangle^{\frac{\alpha'(1-\beta)}{2} - 2} \mathbb{1}_{|\xi|<1} + \mathbb{1}_{|\xi|\geq 1} \langle v \rangle^{-2},$$

If $\alpha' < 2$, then $\frac{\alpha'(1-\beta)}{2} - 2 = \frac{\mathfrak{a}}{2} - 2 \leq 0$ and if $\alpha' = 2$, then $\frac{\alpha'(1-\beta)}{2} - 2 = -1 - \beta \leq 0$, therefore, the right hand side of the above equation is decreasing with respect to $|v|$, which yields formula (5.16).

- ***Inequality* (5.17)*: the Hölder bound***
  Assume $\beta \leq 1$ and $|\xi| \leq 1$ and let $\ell = \frac{\alpha'(1-\beta)}{2} \in (0,1)$. In this case, we want to use the first estimate in Lemma 5.19. We thus look at $m(v) - m(u)$ where we can assume without loss of generality that $|u| \leq |v|$. First, consider the case when $|u| \leq |u - v|$. Then since

$$|m(v)| \leq |\xi| \langle v \rangle \varphi \leq |\xi|^{\frac{\alpha'}{2}} \langle v \rangle^{\frac{\alpha'(1-\beta)}{2}} \lesssim |\xi|^{\frac{\alpha'}{2}} |v|^{\ell},$$

we deduce

$$\begin{aligned} |m(v) - m(u)| &\lesssim |\xi|^{\frac{\alpha'}{2}} (|v|^{\ell} + |u|^{\ell}) \lesssim |\xi|^{\frac{\alpha'}{2}} (|u-v|^{\ell} + 2|u|^{\ell}) \\ &\lesssim |\xi|^{\frac{\alpha'}{2}} |u - v|^{\ell}. \end{aligned}$$

Then consider the case when $|u| > |u-v|$. Since $|\nabla_v \varphi| \lesssim \langle v\rangle^{-1} \varphi$, we obtain

$$|\nabla m(z)| \lesssim |\xi|\, \varphi(z) \lesssim |\xi|^{\frac{\alpha'}{2}} \langle z\rangle^{\ell-1}.$$

Since $\ell \leq 1$, this bound is a decreasing function of $|z|$. Therefore, since $|u| \leq |v|$, we obtain

$$\begin{aligned} |m(v) - m(u)| &\leq \sup_{|u|\leq|z|\leq|v|} |\nabla m(z)||u-v| \\ &\lesssim |\xi|^{\frac{\alpha'}{2}} \langle u\rangle^{\ell-1} |u-v|. \end{aligned}$$

Then, using the fact that $|u-v| < |u| \leq \langle u\rangle$, we deduce

$$|m(v) - m(u)| \lesssim |\xi|^{\frac{\alpha'}{2}} |u-v|^{\ell}.$$

This finishes the proof of inequality (5.17).

- ***Inequalities*** (5.18) ***and*** (5.19)

Inequality (5.18) is a direct consequence of the fact that the right hand side of formula (5.20) is a non-increasing function of $|v|$ when $\beta \geq 1$ or $|\xi| \geq 1$. It also yields

$$\int_{B_0(\langle v\rangle)} |m| \lesssim \int_{B_0(|v|)} |\xi|^{\frac{\alpha'}{2}} \mathbb{1}_{|\xi|<1} + \mathbb{1}_{|\xi|\geq 1}\, \mathrm{d}v.$$

which proves (5.19). □

Proof of Proposition 5.18. First, combining the formulas of Lemma 5.20 and using Lemma 5.19 yields the bound (5.15). Then by recalling $F^{-1}\mathsf{L}^*(F\cdot) = \Delta_v^{\frac{\mathfrak{a}}{2}} - E\cdot\nabla_v$, we have

$$\begin{aligned} \mu_{\mathsf{L}}^2 &= \int_{\mathbb{R}^d} \left|\Delta_v^{\frac{\mathfrak{a}}{2}}\left((v\cdot\xi)\varphi(v)\right) - E(v)\cdot\nabla_v\left((v\cdot\xi)\varphi(v)\right)\right|^2 \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\beta}} \\ &\leq 2\int_{\mathbb{R}^d} \left|\Delta_v^{\frac{\mathfrak{a}}{2}}\left((v\cdot\xi)\varphi(v)\right)\right|^2 \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\beta}} + 2\int_{\mathbb{R}^d} \left|E(v)\cdot\nabla_v\left((v\cdot\xi)\varphi(v)\right)\right|^2 \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\beta}}. \end{aligned}$$

The first integral is controlled by (5.15). The second integral is estimated as follows. Recalling Formula (5.4), we obtain

$$\begin{aligned} |E(v)\cdot\nabla_v((v\cdot\xi)\varphi)| &\lesssim |E(v)\cdot\xi\varphi| + |(v\cdot\xi)E(v)\cdot\nabla_v\varphi| \\ &\lesssim |v\cdot\xi|\,\langle v\rangle^{\beta}\varphi + |v\cdot\xi|\varphi\,\langle v\rangle^{-2}\langle v\rangle^{\beta}|v|^2 \\ &\lesssim |v\cdot\xi|\,\langle v\rangle^{\beta}\varphi, \end{aligned}$$

so that

$$\|E(v)\cdot\nabla_v((v\cdot\xi)\varphi)\|^2_{L^2(\langle v\rangle^{-\beta}F\,\mathrm{d}v)} \lesssim \int_{\mathbb{R}^d} \frac{|v\cdot\xi|^2}{\left(1+\langle v\rangle^{2|1-\beta|}|\xi|^2\right)^2}\frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\beta}}.$$

Then we obtain

$$\left(\int_{\mathbb{R}^d} \frac{|v\cdot\xi|^2}{\left(1+\langle v\rangle^{2|1-\beta|}|\xi|^2\right)^2}\frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\beta}}\right)^{\frac{1}{2}} \underset{\xi\to 0}{\sim} |\xi|^{\min\left(1,1+\frac{\gamma+\beta-2}{2|1-\beta|}\right)}$$
$$\underset{\xi\to\infty}{\sim} |\xi|^{-1},$$

from which we deduce the result. □

Proposition 5.21. *Let $\gamma \geq |\beta|$. Then there exists a constant $C > 0$ independent from ξ such that*

$$\lambda_{\mathsf{L}} \leq C.$$

As previously we first prove a basic bound on the fractional Laplacian of the weight function.

Lemma 5.22. *For all $v \in \mathbb{R}^d$,*

$$\left|\Delta_v^{\frac{\mathfrak{a}}{2}}\varphi_0(\xi,v)\right| \lesssim \frac{|\xi|^{\mathfrak{a}}}{\langle\xi\rangle^{2+\mathfrak{a}}}.$$

Proof of Lemma 5.22. We use the homogeneity of the fractional Laplacian to get

$$\begin{aligned}\Delta_v^{\frac{\mathfrak{a}}{2}}\varphi_0 &= \Delta_v^{\frac{\mathfrak{a}}{2}}\left(\frac{1}{\langle\xi\rangle^2+|v|^2|\xi|^2}\right)\\ &= \langle\xi\rangle^{-2}\Delta_v^{\frac{\mathfrak{a}}{2}}\left(\left(1+\left|v\langle\xi\rangle^{-1}|\xi|\right|^2\right)^{-1}\right)\\ &= |\xi|^{\mathfrak{a}}\langle\xi\rangle^{-2-\mathfrak{a}}\Delta_v^{\frac{\mathfrak{a}}{2}}\left(\langle v\rangle^{-2}\right)\left(\langle\xi\rangle^{-1}|\xi|v\right).\end{aligned}$$

The result follows from the fact that $\Delta_v^{\frac{\mathfrak{a}}{2}}\left(\langle v\rangle^{-2}\right)$ is a bounded function by Lemma 5.19. □

Proof of Proposition 5.21. We follow the same steps to estimate λ_{L}. We have

$$\begin{aligned}\|\varphi_0 F\|^2\lambda_{\mathsf{L}}(\xi)^2 &= \int_{\mathbb{R}^d}\left(\Delta_v^{\frac{\mathfrak{a}}{2}}(\varphi_0(v)) - E(v)\cdot\nabla_v(\varphi_0(v))\right)^2\frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\beta}}\\ &\leq 2\int_{\mathbb{R}^d}\left|\Delta_v^{\frac{\mathfrak{a}}{2}}(\varphi_0(v))\right|^2\frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\beta}} + 2\int_{\mathbb{R}^d}|E(v)\cdot\nabla_v(\varphi_0(v))|^2\frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\beta}}.\end{aligned}$$

We now estimate both integrals of the last right hand side separately. Using Lemma 5.22 above, since $\gamma+\beta > 0$, we get

$$\int_{\mathbb{R}^d}\left|\Delta_v^{\frac{\mathfrak{a}}{2}}(\varphi_0(v))\right|^2\frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\beta}} \lesssim \langle\xi\rangle^{-4}.$$

For the second integral, recalling that $\varphi_0 = \frac{1}{1+\langle v\rangle^2|\xi|^2}$ yields $\nabla_v\varphi_0 = -2\,v\,|\xi|^2\,\varphi_0^2$ and thus by Formula 5.4, we can estimate

$$|E(v)\cdot\nabla_v\varphi_0| \lesssim \langle v\rangle^\beta\,|v|^2|\xi|^2\varphi_0^2 \lesssim \langle v\rangle^\beta\,\varphi_0.$$

Consequently,

$$\int_{\mathbb{R}^d} |E(v)\;\cdot\nabla_v(\varphi_0(v))|^2\,\frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\beta}} \le \int_{\mathbb{R}^d} \frac{1}{\left(1+\langle v\rangle^2\,|\xi|^2\right)^2}\frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma-\beta}} \lesssim \langle\xi\rangle^{-4}\,.$$

Since $\|\varphi_0 F\|^{-2} \le \langle\xi\rangle^4$, λ_{L} is bounded. □

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Appendix B

Weighted Poincaré and polynomial rate of convergence

B.1 A twisted Poincaré inequality

This appendix is devoted to a proof of (4.4), (5.13) and considerations on related inequalities. First, considering the case of inequality (5.13), we have the following proof inspired from [41]. See also [64].

Proposition B.23. *Let $\gamma > 2$ and $F(v) = \frac{C_\gamma}{\langle v\rangle^{d+\gamma}}$. Then there exists a constant $\mathcal{C}_P > 0$ such that*

$$\int_{\mathbb{R}^d} |h - \langle h\rangle_F|^2 \langle v\rangle^{-2} F \,\mathrm{d}v \le \mathcal{C}_P \int_{\mathbb{R}^d} |\nabla_v h|^2 F \,\mathrm{d}v.$$

Proof. Assume $\langle h\rangle_F = 0$. Setting $g = hF^a$ with $a = \frac{\gamma+2}{2(d+\gamma)}$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} |\nabla_v h|^2 F \,\mathrm{d}v &= \int_{\mathbb{R}^d} |\nabla_v g|^2 F^{1-2a} + |g|^2 |\nabla_v F^{-a}|^2 F + \frac{a}{2a-1} \nabla_v(g^2)\cdot\nabla_v(F^{-2a}) \,\mathrm{d}v \\
&\ge \int_{\mathbb{R}^d} |h|^2 \left(\left| \frac{\nabla_v (F^{-a})}{F^{-a}} \right|^2 + \frac{a}{1-2a} \frac{\Delta_v (F^{-2a})}{F^{-2a}} \right) F \,\mathrm{d}v \\
&\ge \frac{\gamma+2}{4} \int_{\mathbb{R}^d} |h|^2 \Big((\gamma+2)|v|^2 - 2d \Big) \langle v\rangle^{-4} F \,\mathrm{d}v \\
&\ge C_{d,\gamma} \int_{\mathbb{R}^d} |h|^2 \langle v\rangle^{-2} F \,\mathrm{d}v - C'_{d,\gamma} \int_{B_R} |h|^2 F \,\mathrm{d}v,
\end{aligned}$$

for a given $R > 0$. We deduce the existence of $C_{d,\gamma}, C'_{d,\gamma} > 0$ such that

$$\int_{\mathbb{R}^d} |h|^2 \langle v\rangle^{-2} F \,\mathrm{d}v \le C_{d,\gamma} \int_{\mathbb{R}^d} |\nabla_v h|^2 F \,\mathrm{d}v + C'_{d,\gamma} \int_{B_R} |h|^2 F \,\mathrm{d}v. \tag{B.1}$$

Then, by the local Poincaré-Wirtinger inequality, we obtain

$$\int_{B_R} |h|^2 F \,\mathrm{d}v \le C_R \int_{B_R} |\nabla_v h|^2 F \,\mathrm{d}v + \frac{1}{\langle 1_{B_R}\rangle_F^2} \left(\int_{B_R} hF \,\mathrm{d}v \right)^2. \tag{B.2}$$

But since $\langle h \rangle_F = 0$, we can write

$$\begin{aligned}\left(\int_{B_R} hF\,\mathrm{d}v\right)^2 &= \left(\int_{B_R^c} hF\,\mathrm{d}v\right)^2 \\ &\leq \left(\int_{B_R^c} \langle v\rangle^2 F\,\mathrm{d}v\right)\left(\int_{B_R^c} |h|^2 \langle v\rangle^{-2} F\,\mathrm{d}v\right) \\ &\leq \varepsilon_R \int_{B_R^c} |h|^2 \langle v\rangle^{-2} F\,\mathrm{d}v,\end{aligned} \tag{B.3}$$

with $\varepsilon_R \underset{R\to\infty}{\to} 0$ if $|v|^2 F \in L^1$. Combining (B.1), (B.2) and (B.3) yields

$$\left(1 - \frac{\varepsilon_R}{\langle 1_{B_R}\rangle_F^2}\right) \int_{B_R^c} |h|^2 \langle v\rangle^{-2} F\,\mathrm{d}v \leq (C_{d,\gamma} + C'_{d,\gamma} C_R) \int_{\mathbb{R}^d} |\nabla_v h|^2 F\,\mathrm{d}v,$$

which proves the result. □

A very similar proof holds for (4.4) and gives the following result

Proposition B.24. *Let $\gamma > 0$ and $F(v) = C_\gamma e^{\langle v\rangle^\gamma}$. Then there exists a constant $\mathcal{C}_P > 0$ such that*

$$\int_{\mathbb{R}^d} |h - \langle h\rangle_F|^2 \langle v\rangle^{2(\gamma-1)} F\,\mathrm{d}v \leq \mathcal{C}_P \int_{\mathbb{R}^d} |\nabla_v h|^2 F\,\mathrm{d}v.$$

B.1.1 Continuous spectrum and weighted Poincaré inequality

Let us consider two probability measures on $\mathbb{R}^d$

$$\mathrm{d}\mu = e^{-\phi}\,\mathrm{d}v \quad \text{and} \quad \mathrm{d}\nu = \psi\,\mathrm{d}\mu,$$

where ϕ and $\psi \geq 0$ are two measurable functions, and the *weighted Poincaré inequality*

$$\forall\, h \in \mathcal{D}(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} |\nabla h|^2\,\mathrm{d}\mu \geq \mathcal{C}_\star \int_{\mathbb{R}^d} |h - \langle h\rangle_\nu|^2\,\mathrm{d}\nu, \tag{B.4}$$

where $\langle h\rangle_\nu = \int_{\mathbb{R}^d} h\,\mathrm{d}\nu$. The question we address here is: *on which conditions on ϕ and ψ do we know that* (B.4) *holds ?* Our key example is

$$\phi(v) = \langle v\rangle^\gamma + \log C_\gamma \quad \text{and} \quad \psi(v) = c_{\gamma,\beta}^{-1} \langle v\rangle^\beta \tag{B.5}$$

with $\gamma > 0$, $\beta < 0$, $C_\gamma = \int_{\mathbb{R}^d} e^{-\phi}\,\mathrm{d}v$ and $c_{\gamma,\beta} = \int_{\mathbb{R}^d} \langle v\rangle^\beta\,\mathrm{d}\mu$.

Let us consider a potential Φ on $\mathbb{R}^d$. We assume that Φ is a measurable function such that

$$\sigma = \lim_{r\to+\infty} \inf_{w\in\mathcal{D}(B_r^c)\setminus\{0\}} \frac{\int_{\mathbb{R}^d} (|\nabla w|^2 + \Phi\,|w|^2)\,\mathrm{d}v}{\int_{\mathbb{R}^d} |w|^2\,\mathrm{d}v} > 0\,,$$

where $B_r^c := \left\{v \in \mathbb{R}^d \,:\, |v| > r\right\}$ and $\mathcal{D}(B_r^c)$ denotes the space of smooth functions on $\mathbb{R}^d$ with compact support in B_r^c. According to Persson's result [178, Theorem 2.1], the lower end σ of the continuous spectrum of the Schrödinger operator $-\Delta + \Phi$ is such that

$$\sigma \geq \lim_{r\to+\infty} \operatorname*{infess}_{v\in B_r^c} \Phi(v) =: \sigma_0.$$

If we replace $\int_{\mathbb{R}^d}|w|^2\,\mathrm{d}v$ by the weighted integral $\int_{\mathbb{R}^d}|w|^2\,\psi\,\mathrm{d}v$ for some measurable function ψ, we have the modified result that the operator $\mathcal{L}=\psi^{-1}\,(-\,\Delta+\Phi)$ on $L^2(\mathbb{R}^d,\psi\,\mathrm{d}v)$, associated with the quadratic form

$$w\mapsto\int_{\mathbb{R}^d}\left(|\nabla w|^2+\Phi\,|w|^2\right)\mathrm{d}v$$

has only discrete eigenvalues in the interval $(-\infty,\sigma)$ where

$$\sigma=\lim_{r\to+\infty}\inf\nolimits_{w\in\mathcal{D}(B_r^c)\setminus\{0\}}\frac{\int_{\mathbb{R}^d}\left(|\nabla w|^2+\Phi\,|w|^2\right)\mathrm{d}v}{\int_{\mathbb{R}^d}|w|^2\,\psi\,\mathrm{d}v}>0.$$

To prove it, it is enough to observe that 0 is the lower end of the continuous spectrum of $\mathcal{L}-\sigma$ by applying again [178, Theorem 2.1]. It is also straightforward to check that the lower end of the continuous spectrum of $\mathcal{L}$ is such that

$$\sigma\geq\lim_{r\to+\infty}\operatorname*{infess}_{v\in B_r^c}\frac{\Phi(v)}{\psi(v)}=:\sigma_0.$$

Notice that σ_0 is either finite or infinite. In the case of (B.5), we get that $\sigma_0\in(0,+\infty]$ if and only if $\beta\leq 2\,(\gamma-1)$. Relating the weighted Poincaré inequality with the spectrum of $\mathcal{L}$ is indeed classical. Let

$$h=w\,e^{\phi/2}\,,\quad\Phi=\tfrac14\,|\nabla\phi|^2-\tfrac12\,\Delta\phi,\tag{B.6}$$

and observe that

$$\int_{\mathbb{R}^d}|\nabla h|^2\,\mathrm{d}\mu=C_\gamma^{-1}\int_{\mathbb{R}^d}\left(|\nabla w|^2+\Phi\,|w|^2\right)\mathrm{d}v\,,\quad\int_{\mathbb{R}^d}\left|h-\langle h\rangle_\nu\right|^2\mathrm{d}\mu=C_\gamma^{-1}\int_{\mathbb{R}^d}|w-\widetilde{w}|^2\;\psi\,\mathrm{d}v\,,$$

where $\widetilde{w}=\frac{\int_{\mathbb{R}^d}w\,\psi\,e^{-\phi/2}\,\mathrm{d}v}{\int_{\mathbb{R}^d}\psi\,e^{-\phi}\,\mathrm{d}v}\,e^{-\phi/2}$.

Proposition B.25. *Let Φ and ψ be two measurable functions such that $\sigma_0>0$. Then Inequality (B.4) holds for some positive, finite, optimal constant $\mathcal{C}_\star>0$, which is at most equal to σ. Otherwise, if we have that $\lim_{r\to+\infty}\operatorname{supess}_{v\in B_r^c}\frac{\Phi(v)}{\psi(v)}=0$, then Inequality (B.4) does not hold.*

Proof. By construction, σ is nonnegative and the infimum of the Rayleigh quotient

$$w\mapsto\frac{\int_{\mathbb{R}^d}\left(|\nabla w|^2+\Phi\,|w|^2\right)\mathrm{d}v}{\int_{\mathbb{R}^d}|w|^2\,\psi\,\mathrm{d}v}$$

is achieved by $h\equiv\langle h\rangle_\nu=1$, that is, $\widetilde{w}=e^{-\phi/2}$, which moreover generates the kernel of $\mathcal{L}$. Hence we can interpret $\mathcal{C}_\star$ as the first positive eigenvalue, if there is any in the interval $(0,\sigma)$, or $\mathcal{C}_\star=\sigma$ if there is none. □

In the case of (B.5), the condition $\beta\geq 2\,(\gamma-1)$ is a necessary and sufficient condition for the inequality (B.4) to hold. The threshold case is remarkable: up to a redefinition of $\mathcal{C}_\star$, for any $\gamma\in(0,1)$, inequality (B.4) can be rewritten as

$$\forall\,h\in\mathcal{D}(\mathbb{R}^d)\,,\quad\int_{\mathbb{R}^d}|\nabla h|^2\,e^{-\langle v\rangle^\gamma}\,\mathrm{d}v\geq\mathcal{C}_\star\int_{\mathbb{R}^d}\frac{\left|h-\langle h\rangle_\nu\right|^2\,e^{-\langle v\rangle^\gamma}}{\left(1+|v|^2\right)^{1-\gamma}}\,\mathrm{d}v,\tag{B.7}$$

for some constant $\mathcal{C}_\star\in\left(0,\gamma^2/4\right]$ and $\langle h\rangle_\nu:=c_\gamma^{-1}\int_{\mathbb{R}^d}\frac{h\,e^{-\langle v\rangle^\gamma}}{(1+|v|^2)^{1-\gamma}}\,\mathrm{d}v$ with $c_\gamma=\int_{\mathbb{R}^d}\frac{e^{-\langle v\rangle^\gamma}}{(1+|v|^2)^{1-\gamma}}\,\mathrm{d}v$.

B.1.2 Twisted Poincaré inequality

In some cases, one can use the above weighted Poincaré inequality (B.4) to prove the twisted Poincaré inequality (4.4). In these cases, we obtain a good estimation of the optimal constant.

Corollary B.26. *Let Φ and ψ^{-1} be respectively a measurable function and a bounded function such that, with the above notations, $\sigma_0 > 0$. Then the inequality*

$$\forall\, h \in \mathcal{D}(\mathbb{R}^d)\,, \quad \int_{\mathbb{R}^d} |\nabla h|^2 \,\mathrm{d}\mu \geq \mathcal{C} \int_{\mathbb{R}^d} \left|h - \langle h \rangle_\mu\right|^2 \mathrm{d}\nu \tag{B.8}$$

holds for some optimal constant $\mathcal{C}_\star > 0$, where $\langle h \rangle_\mu := \int_{\mathbb{R}^d} h \,\mathrm{d}\mu$ denotes the average with respect to the measure $\mathrm{d}\mu$. Moreover, with the notation $\mathcal{C}_\star$ of Proposition B.25, we have

$$\mathcal{C}_\star \leq \mathcal{C} \leq \left(1 + \left\|\psi^{-1} - 1\right\|_\infty^2\right) \mathcal{C}_\star.$$

In case $\phi(v) = \langle v \rangle^\gamma$, Inequality (B.8) is equivalent to [130, inequality (1.12)], which can be deduced from [16]: see Appendix B.2.1 for more details.

Proof. Let us notice that

$$\int_{\mathbb{R}^d} |h - c\,|^2 \,\mathrm{d}\nu = \int_{\mathbb{R}^d} h^2 \,\mathrm{d}\nu - 2\,c \int_{\mathbb{R}^d} h \,\mathrm{d}\nu + c^2 \geq \int_{\mathbb{R}^d} |h - \langle h \rangle_\nu|^2 \,\mathrm{d}\nu,$$

with equality if and only if $c = \langle h \rangle_\nu$ in the left hand side. Reciprocally, we compute

$$\int_{\mathbb{R}^d} \left|h - \langle h \rangle_\mu\right|^2 \mathrm{d}\nu = \int_{\mathbb{R}^d} |h - \langle h \rangle_\nu + \langle h \rangle_\nu - \langle h \rangle_\mu\,|^2 \,\mathrm{d}\nu = \int_{\mathbb{R}^d} |h - \langle h \rangle_\nu|^2 \,\mathrm{d}\nu + |\,\langle h \rangle_\nu - \langle h \rangle_\mu\,|^2$$

and, using the fact that $\langle h \rangle_\nu = \int_{\mathbb{R}^d} \langle h \rangle_\nu \,\mathrm{d}\nu = \int_{\mathbb{R}^d} \langle h \rangle_\nu \,\mathrm{d}\mu = \int_{\mathbb{R}^d} \langle h \rangle_\nu \,\psi^{-1} \,\mathrm{d}\nu$ because $\mathrm{d}\mu$ and $\mathrm{d}\nu$ are two probability measures,

$$\begin{aligned} \left|\langle h \rangle_\nu - \langle h \rangle_\mu\right|^2 = \left(\int_{\mathbb{R}^d} h \,\mathrm{d}\nu - \int_{\mathbb{R}^d} h \,\mathrm{d}\mu\right)^2 &= \left(\int_{\mathbb{R}^d} (1 - \psi^{-1})\,(h - \langle h \rangle_\mu)\,\mathrm{d}\nu\right)^2 \\ &\leq \left\|\psi^{-1} - 1\right\|_\infty^2 \int_{\mathbb{R}^d} \left|h - \langle h \rangle_\mu\right|^2 \mathrm{d}\nu. \end{aligned}$$

Altogether, we have that

$$\int_{\mathbb{R}^d} |h - \langle h \rangle_\nu|^2 \,\mathrm{d}\nu \leq \int_{\mathbb{R}^d} \left|h - \langle h \rangle_\mu\right|^2 \mathrm{d}\nu \leq \left(1 + \left\|\psi^{-1} - 1\right\|_\infty^2\right) \int_{\mathbb{R}^d} |h - \langle h \rangle_\nu|^2 \,\mathrm{d}\nu.$$

□

However, in the case of (B.5), we have $\|\psi^{-1} - 1\|_\infty = \infty$, so that the above proof does not apply in this context.

B.2 Algebraic decay rates for the Fokker-Planck equation

Here we consider simple estimates of the decay rates for the homogeneous case given by $f(t,x,v) = g(t,v)$ of Equation (4.1), that is, the Fokker-Planck equation

$$\partial_t g = \mathsf{L}_1 g \tag{B.9}$$

with equilibrium $F(v) = C_\gamma e^{-\langle v\rangle^{-\gamma}}$.

B.2.1 Weak Poincaré inequality

We assume that $\gamma \in (0,1)$ and $\beta \in (2(\gamma-1),0)$. By a simple Hölder inequality, with $p+1 = 2(1-\gamma)/|\beta|$, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^d} \left|h - \langle h\rangle_\mu\right|^2 \mathrm{d}\mu &= \int_{\mathbb{R}^d} \left|h - \langle h\rangle_\mu\right|^2 \langle v\rangle^\beta \langle v\rangle^{-\beta}\,\mathrm{d}\mu \\ &\le \left(\int_{\mathbb{R}^d} \left|h - \langle h\rangle_\mu\right|^2 \langle v\rangle^{2(\gamma-1)}\,\mathrm{d}\mu\right)^{\frac{1}{p+1}} \left(\int_{\mathbb{R}^d} \left\|h - \langle h\rangle_\mu\right\|^2_{L^\infty(\mathbb{R}^d)} \langle v\rangle^{\frac{2}{p}(1-\gamma)}\,\mathrm{d}\mu\right)^{\frac{p}{p+1}}.\end{aligned}$$

Here we choose $\langle h\rangle_\mu := \int_{\mathbb{R}^d} h\,\mathrm{d}\mu$. Using (4.4), we end up with

$$\forall\, h \in \mathcal{D}(\mathbb{R}^d)\,,\quad \int_{\mathbb{R}^d} \left|h - \langle h\rangle_\mu\right|^2 \mathrm{d}\mu \le \mathcal{C}_{\gamma,p} \left(\int_{\mathbb{R}^d} |\nabla h|^2\,\mathrm{d}\mu\right)^{\frac{1}{p+1}} \left\|h - \langle h\rangle_\mu\right\|^{\frac{2p}{p+1}}_{L^\infty(\mathbb{R}^d)}\,, \tag{B.10}$$

for some explicit positive constant $\mathcal{C}_{\gamma,p}$. We learn from (4.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \left|h(t,\cdot) - \langle h\rangle_\mu\right|^2 \mathrm{d}\mu = -\,2 \int_{\mathbb{R}^d} |\nabla_v h|^2\,\mathrm{d}\mu$$

if $g = h\,F$ is solves (B.9), and we also know that $\langle h\rangle_\mu$ does not depend on t. By a strategy that goes back at least to [149, Theorem 2.2] and is due, according to the author, to D. Stroock, we obtain that

$$\int_{\mathbb{R}^d} \left|h(t,\cdot) - \langle h\rangle_\mu\right|^2 \mathrm{d}\mu \le \left(\left(\int_{\mathbb{R}^d} \left|h(0,\cdot) - \langle h\rangle_\mu\right|^2 \mathrm{d}\mu\right)^{-p} + \frac{2\,p}{\mathcal{C}_{\gamma,p}^{p+1}\,\mathcal{M}}\,t\right)^{-\frac{1}{p}},$$

with $\mathcal{M} = \sup_{s\in(0,t)} \left\|h(s,\cdot) - \langle h\rangle_\mu\right\|^{2p}_{L^\infty(\mathbb{R}^d)}$. The limitation is of course that we need to restrict the initial conditions in order to have $\mathcal{M}$ uniformly bounded with respect to t. Since β can be chosen arbitrarily close to $2(\gamma-1)$, the exponent $1/p$ can be taken arbitrarily large but to the price of a constant $\mathcal{C}_{\gamma,p}$ which explodes as $\beta \to 2(\gamma-1)$.

Notice that (B.10) is equivalent to the *weak Poincaré inequality*

$$\forall\, h \in \mathcal{D}(\mathbb{R}^d)\,,\quad \mathcal{C}_{\gamma,p}^{-1} \int_{\mathbb{R}^d} \left|h - \langle h\rangle_\mu\right|^2 \mathrm{d}\mu \le (p+1)\,r^{-\frac{1}{p+1}} \int_{\mathbb{R}^d} |\nabla h|^2\,\mathrm{d}\mu + r\,\left\|h - \langle h\rangle_\mu\right\|^2_{L^\infty(\mathbb{R}^d)}\,,$$

for all $r > 0$, as stated in [184, (1.6) and Example 1.4 (c)]. The equivalence of this inequality and (B.10) is easily recovered by optimizing on $r > 0$. It is worth to remark that here we consider $\left\|h - \langle h\rangle_\mu\right\|_{L^\infty(\mathbb{R}^d)}$ while various other quantities like, *e.g.*, the median can be used in weak Poincaré inequalities.

B.2.2 Weighted L^2 estimates

As an alternative to the *weak Poincaré inequality* approach of Appendix B.2.1, we can consider for some arbitrary $k > 0$ the evolution according to Equation (B.9) of $\int_{\mathbb{R}^d} |h(t,v)|^2 \langle v\rangle^k \,\mathrm{d}\mu = \int_{\mathbb{R}^d} |h(t,v)|^2 \langle v\rangle^k F \,\mathrm{d}v$ where $h := g/F$ solves

$$\partial_t h = F^{-1}\, \nabla_v \cdot \Big(F\, \nabla_v h\Big).$$

Let us compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |h(t,v)|^2 \,\langle v\rangle^k\, F \,\mathrm{d}v + 2\int_{\mathbb{R}^d} |\nabla_v h|^2\, \langle v\rangle^k\, F\,\mathrm{d}v = -\int_{\mathbb{R}^d} \nabla_v(h^2)\cdot \nabla_v\left(\langle v\rangle^k\right) F\,\mathrm{d}v,$$

and observe with $\ell = \gamma - 2$ that

$$\nabla_v \log\Big(F\,\nabla_v\big(\langle v\rangle^k\big)\Big) = \frac{k}{\langle v\rangle^4}\left(d + (k+d-2)\,|v|^2 - \gamma\,\langle v\rangle^\gamma\,|v|^2\right) \le a - b\,\langle v\rangle^{-\ell},$$

for some $a \in \mathbb{R}$, $b \in (0,+\infty)$. The same proof as in Proposition 4.3 shows that there exists a constant $\mathcal{K}_k > 0$ such that

$$\forall\, t \ge 0 \qquad \|h(t,\cdot)\|_{L^2\left(\langle v\rangle^k \mathrm{d}\mu\right)} \le \mathcal{K}_k\, \left\|h^{\mathrm{in}}\right\|_{L^2\left(\langle v\rangle^k\,\mathrm{d}\mu\right)}.$$

Hence, if h solves (B.9) with initial value h^{in}, we can use (4.4) to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\left|h(t,\cdot) - \langle h\rangle_\mu\right|^2 \mathrm{d}\mu = -\,2\int_{\mathbb{R}^d} |\nabla_v h|^2\,\mathrm{d}\mu \le -\,2\,\mathcal{C}\int_{\mathbb{R}^d}\left|h - \langle h\rangle_\mu\right|^2\,\langle v\rangle^{2\,(\gamma-1)}\,\mathrm{d}\mu,$$

and Hölder's inequality with $\theta = k/\Big(k + 2\,(1-\gamma)\Big)$ to estimate the right hand side

$$\int_{\mathbb{R}^d}\left|h - \langle h\rangle_\mu\right|^2 \mathrm{d}\mu \le \left(\int_{\mathbb{R}^d}\left|h - \langle h\rangle_\mu\right|^2\,\langle v\rangle^{2\,(\gamma-1)}\,\mathrm{d}\mu\right)^\theta \left(\int_{\mathbb{R}^d}\left|h - \langle h\rangle_\mu\right|^2\,\langle v\rangle^k\,\mathrm{d}\mu\right)^{1-\theta}.$$

Proposition B.27. *Let $g^{\mathrm{in}} \in L^1(\mathrm{d}\mu) \cap L^2(\langle v\rangle^k\,\mathrm{d}\mu)$ for some $k > 0$ and consider the solution g to* (B.9) *with initial datum g^{in}. With $\mathcal{C}$ as in* (4.4)*, if $\overline{g} = (\int_{\mathbb{R}^d} g\,\mathrm{d}v)\,F$, then*

$$\int_{\mathbb{R}^d} |g(t,\cdot) - \overline{g}|^2\,\mathrm{d}\mu \le \left(\left(\int_{\mathbb{R}^d}\left|g^{\mathrm{in}} - \overline{g}\right|^2 \mathrm{d}\mu\right)^{-\eta} + 4\,\frac{\eta\,\mathcal{C}}{\mathcal{K}^\eta}\,t\right)^{-1/\eta},$$

with $\eta = 2\,(1-\gamma)/k$ and $\mathcal{K} := \mathcal{K}_k\,\|g^{\mathrm{in}}\|^2_{L^2\left(\langle v\rangle^k\,\mathrm{d}\mu\right)} + \Theta_k\,(\int_{\mathbb{R}^d} g^{\mathrm{in}}\,\mathrm{d}v)^2$.

We notice that arbitrarily large decay rates can be obtained under the condition that $k > 0$ is large enough. With $\beta = 2\,(\gamma-1)$, we recover that when $k < d|\beta|/2$, the rate of relaxation to the equilibrium is slower than $(1+t)^{-d/2}$ as induced by the result of Proposition B.27 and responsible for the limitation that appears in Theorem 4.1.

Proof. We recall that $g = h\,F$, $\overline{g} = \langle h\rangle_\mu\,F$ and $F\,\mathrm{d}\mu = \mathrm{d}v$. Using

$$\frac{1}{2}\int_{\mathbb{R}^d}\left|h - \langle h\rangle_\mu\right|^2\,\langle v\rangle^k\,\mathrm{d}\mu \le \int_{\mathbb{R}^d} |h|^2\,\langle v\rangle^k\,\mathrm{d}\mu + \Theta_k\,\langle h\rangle_\mu^2 = \mathcal{K},$$

we obtain that $y(t) := \int_{\mathbb{R}^d} |g(t,\cdot) - \overline{g}|^2\,\mathrm{d}\mu$ obeys to $y' \le -\,2\,\mathcal{C}\,\mathcal{K}^{1-1/\theta}\,y^{1/\theta}$ and conclude by a Grönwall estimate. □

Appendix C

Fractional Laplacian with drift

C.1 Steady states and force field

In this section, we consider the case when $F(v) = C_\gamma \langle v \rangle^{-(d+\gamma)}$ and $\mathsf{L} = \Delta_v^{\frac{\mathfrak{a}}{2}} + \nabla_v \cdot (E\cdot)$ so that by definition we get

$$\nabla_v \cdot (EF) = (-\Delta_v)^{\frac{\mathfrak{a}}{2}} F = -\left(\nabla_v \cdot \left(\nabla_v((-\Delta_v)^{\frac{\mathfrak{a}-2}{2}} F)\right)\right),$$

If E is radial, this implies

$$E(v)F(v) = -\nabla_v((-\Delta_v)^{\frac{\mathfrak{a}-2}{2}} F) = -C_{d,\gamma,\mathfrak{a}} \nabla_v \left(\frac{1}{|v|^{d+\mathfrak{a}-2}} * \frac{1}{\langle v \rangle^{d+\gamma}} \right).$$

From this expression we deduce the following result.

Proposition C.28. *Assume $\mathfrak{a} \in (0,2)$. Then*

$$E(v) = G(v) \langle v \rangle^{\beta} v$$

where $G \in L^\infty(\mathbb{R}^d)$ is a positive function such that $G^{-1} \in L^\infty(B_0^c(1))$.

Proof. Let $u(v) = -\nabla_v \left(\frac{1}{|v|^{d+\mathfrak{a}-2}} * \frac{1}{\langle v \rangle^{d+\gamma}} \right)$. We have

$$u(v) = (d+\gamma) \left(\frac{1}{|v|^{d+\mathfrak{a}-2}} * \frac{v}{\langle v \rangle^{d+\gamma+2}} \right),$$

so that since $\frac{v}{\langle v \rangle^{d+\gamma+2}} \in C^\infty \cap L^1$ and $\mathfrak{a} < 2$, $u \in L^\infty_{\text{loc}}$. Now we assume that $\mathfrak{a} \in (0,1)$ and $|v| > 1$ and we write

$$u(v) = (d+\mathfrak{a}-2) \left(\frac{v}{|v|^{d+\mathfrak{a}}} * \frac{1}{\langle v \rangle^{d+\gamma}} \right). \tag{C.1}$$

Then since

$$\int_{|v'|\geq \frac{|v|}{2}} \frac{v'}{|v'|^{d+\mathfrak{a}}} \frac{\mathrm{d}v'}{\langle v'-v\rangle^{d+\gamma}} \leq \left(\int_{\mathbb{R}^d} \frac{\mathrm{d}v'}{\langle v'\rangle^{d+\gamma}}\right) \frac{2^{d+\mathfrak{a}-1}}{|v|^{d+\mathfrak{a}-1}}$$
$$\int_{|v'|< \frac{|v|}{2}} \frac{v'}{|v'|^{d+\mathfrak{a}}} \frac{\mathrm{d}v'}{\langle v'-v\rangle^{d+\gamma}} \leq \left(\int_{B_0(|v|/2)} \frac{\mathrm{d}v'}{|v'|^{d+\mathfrak{a}-1}}\right) \frac{2^{d+\mathfrak{a}-1}}{|v|^{d+\gamma}} \leq \frac{2^{d+\mathfrak{a}-1}\omega_d}{1-\mathfrak{a}} \frac{1}{|v|^{d+\gamma+\mathfrak{a}-1}},$$

we deduce that

$$|u(v)\cdot v| \leq |u(v)||v| \leq C|v|^{-(d+\mathfrak{a}-2)}.$$

To get a bound by below on $u(v)\cdot v$, we cut the integral in two parts and we use the fact that $|v|>1$ and $|v'-v|<1/2$ implies $v'\cdot v>0$. First

$$\begin{aligned}\int_{\substack{|v'-v|>1/2\\|v'+v|>1/2}} \frac{v'\cdot v}{|v'|^{d+\mathfrak{a}}} \frac{\mathrm{d}v'}{\langle v'-v\rangle^{d+\gamma}} &= \left(\int_{\substack{|v'-v|>1/2\\ v'\cdot v>0}} + \int_{\substack{|v'+v|>1/2\\ v'\cdot v<0}}\right) \frac{v'\cdot v}{|v'|^{d+\mathfrak{a}}} \frac{\mathrm{d}v'}{\langle v'-v\rangle^{d+\gamma}} \\ &= \int_{\substack{|v'-v|>1/2\\ v'\cdot v>0}} \frac{v'\cdot v}{|v'|^{d+\mathfrak{a}}} \left(\frac{1}{\langle v'-v\rangle^{d+\gamma}} - \frac{1}{\langle v'+v\rangle^{d+\gamma}}\right) \mathrm{d}v',\end{aligned}$$

which is positive since $\langle v'+v\rangle^2 - \langle v'-v\rangle^2 = 2\,v'\cdot v \geq 0$. The remaining terms are treated as follows

$$\begin{aligned}\int_{\substack{|v'-v|\leq 1/2\\ \text{or}\\ |v'+v|\leq 1/2}} \frac{v'\cdot v}{|v'|^{d+\mathfrak{a}}} \frac{\mathrm{d}v'}{\langle v'-v\rangle^{d+\gamma}} &= \int_{|v'-v|<\frac{1}{2}} \frac{v'\cdot v}{|v'|^{d+\mathfrak{a}}} \left(\frac{1}{\langle v'-v\rangle^{d+\gamma}} - \frac{1}{\langle v'+v\rangle^{d+\gamma}}\right) \mathrm{d}v' \\ &\geq \left(\frac{1}{\langle 1/2\rangle^{d+\gamma}} - \frac{1}{\langle 3/2\rangle^{d+\gamma}}\right) \int_{|v'-v|<\frac{1}{2}} \frac{v'\cdot v}{|v'|^{d+\mathfrak{a}}} \mathrm{d}v',\end{aligned}$$

since $|v'+v| \geq 2|v| - |v'-v| \geq \frac{3}{2}$. Finally, since $|v|>1$, when $|v'-v|<\frac{1}{2}$ we get

$$\begin{aligned} 2\,v'\cdot v = |v|^2 + |v'|^2 - |v'-v|^2 \geq |v|^2 - \tfrac{1}{2} \geq \frac{|v|^2}{2} \\ |v'| \leq |v| + |v'-v| \leq 2|v|,\end{aligned}$$

so that

$$\int_{|v'-v|<\frac{1}{2}} \frac{v'\cdot v}{|v'|^{d+\mathfrak{a}}} \mathrm{d}v' \geq \frac{|B_0(1/2)|}{2^{d+\mathfrak{a}+2}} \frac{1}{|v|^{d+\mathfrak{a}-2}}.$$

This implies that $u(v)\cdot v \geq C|v|^{-(d+\mathfrak{a}-2)}$. Therefore, there exists $G(v) := (u(v)\cdot v)\,|v|^{d+\mathfrak{a}-2} \in L^\infty$ and $G^{-1} \in L^\infty(B_0^c(1))$. By remarking that u is radial, we can write

$$u(v) = G(v)\,\frac{v}{|v|^{d+\mathfrak{a}}},$$

and we deduce the result by writing $E(v) = CF(v)^{-1}u(v)$ and using the fact that $\beta = \gamma - \mathfrak{a}$. In the case when $\mathfrak{a} \in [1,2)$, the gradient of $\frac{1}{|v|^{d+\mathfrak{a}-2}}$ is a distribution of order 1 that can be defined as a principal value since in the sense of distributions it holds for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$

$$\begin{aligned}\left\langle \nabla_v \left(\frac{1}{|v|^{d+\mathfrak{a}-2}} \right), \varphi \right\rangle_{\mathcal{D}',\mathcal{D}} &= -\int_{\mathbb{R}^d} \frac{\nabla_v \varphi(v)}{|v|^{d+\mathfrak{a}-2}} \,\mathrm{d}v \\ &= -\int_{\mathbb{R}^d} \frac{\nabla_v (\varphi(v) - \varphi(0))}{|v|^{d+\mathfrak{a}-2}} \,\mathrm{d}v \\ &= (d+\mathfrak{a}-2) \int_{\mathbb{R}^d} \frac{v}{|v|^{d+\mathfrak{a}}} \left(\varphi(v) - \varphi(0)\right) \mathrm{d}v \\ &=: (d+\mathfrak{a}-2) \left\langle \mathrm{pv}\left(\frac{v}{|v|^{d+\mathfrak{a}}} \right), \varphi \right\rangle_{\mathcal{D}',\mathcal{D}}.\end{aligned}$$

We deduce that formula (C.1) becomes

$$\begin{aligned}u(v) &= (d+\mathfrak{a}-2)\,\mathrm{pv}\left(\frac{v}{|v|^{d+\mathfrak{a}}} \right) * \frac{1}{\langle v \rangle^{d+\gamma}} \\ &= (d+\mathfrak{a}-2) \int_{\mathbb{R}^d} \frac{v'}{|v'|^{d+\mathfrak{a}}} \left(\frac{1}{\langle v - v' \rangle^{d+\gamma}} - \frac{1}{\langle v \rangle^{d+\gamma}} \right) \mathrm{d}v,\end{aligned}$$

so that

$$|u(v)| \leq \int_{\mathbb{R}^d} \frac{1}{|v'-v|^{d+\mathfrak{a}-1}} \left| \frac{1}{\langle v' \rangle^{d+\gamma}} - \frac{1}{\langle v \rangle^{d+\gamma}} \right| \mathrm{d}v,$$

which in the same way as in Lemma 5.19, yields

$$|u(v)| \leq \frac{C}{\langle v \rangle^{\min(d+\gamma,d)+\mathfrak{a}-1}} = \frac{C}{\langle v \rangle^{d+\mathfrak{a}-1}}.$$

For $|v| > 1$, we have

$$\begin{aligned}&\int_{|v'|<\frac{1}{2}} \frac{v' \cdot v}{|v'|^{d+\mathfrak{a}}} \left(\frac{1}{\langle v - v' \rangle^{d+\gamma}} - \frac{1}{\langle v \rangle^{d+\gamma}} \right) \mathrm{d}v \\ &\qquad \leq (d+\gamma) \int_{|v'|<\frac{1}{2}} \frac{|v|}{|v'|^{d+\mathfrak{a}-2}} \sup_{w \in B_v(1/2)} \left(\langle w \rangle^{-(d+\gamma+1)} \right) \mathrm{d}v \leq \frac{C}{\langle v \rangle^{d+\gamma}}.\end{aligned}$$

The other part can be treated as in the case $\mathfrak{a} \in (0,1)$ since

$$\int_{|v'|\geq\frac{1}{2}} \frac{v'}{|v'|^{d+\mathfrak{a}}} \left(\frac{1}{\langle v - v' \rangle^{d+\gamma}} - \frac{1}{\langle v \rangle^{d+\gamma}} \right) \mathrm{d}v = \int_{|v'|\geq\frac{1}{2}} \frac{v'}{|v'|^{d+\mathfrak{a}}} \frac{1}{\langle v - v' \rangle^{d+\gamma}} \mathrm{d}v.$$

Therefore, we deduce that

$$\int_{|v'|\geq\frac{1}{2}} \frac{v' \cdot v}{|v'|^{d+\mathfrak{a}}} \left(\frac{1}{\langle v - v' \rangle^{d+\gamma}} - \frac{1}{\langle v \rangle^{d+\gamma}} \right) \mathrm{d}v \geq \frac{C}{\langle v \rangle^{d+\mathfrak{a}-2}}.$$

Since $\mathfrak{a} - 2 < 0 < \gamma$, it yields $d + \gamma > d + \mathfrak{a} - 2$ and we obtain $u(v) \cdot v \geq C|v|^{-(d+\mathfrak{a}-2)}$. As in the case $\mathfrak{a} \in (0,1)$ this yields the result. □

C.2 Poincaré inequality for Fractional Fokker-Planck operators

We take here $\mathsf{L}f = \Delta_v^{\frac{\mathfrak{a}}{2}} f + \nabla_v \cdot (E(v)f)$. Define,

$$\mathfrak{D}_2(f)^2 := \int_{\mathbb{R}^d} \frac{|f(v') - f(v)|^2}{2|v'-v|^{d+\mathfrak{a}}} \mathrm{d}v' = I(|f|^2) - 2f\ I(f).$$

Proposition C.29 (Fractional Poincaré with drift)**.** *Let $\gamma > \mathfrak{a}/2$. There exists $\mathcal{C}_P > 0$ depending on $\mathfrak{a}$ and F such that*

$$-\langle \mathsf{L}f, f\rangle \geq \mathcal{C}_P \left\|(1-\Pi)f\right\|^2_{L^2(\langle v\rangle^{\gamma-\mathfrak{a}} F\,\mathrm{d}v)}. \tag{C.2}$$

Proof. Let $g := fF^{-1}$. Then it holds

$$\begin{aligned} 2\int_{\mathbb{R}^d} (\mathsf{L}f)\, fF^{-1}\,\mathrm{d}v &= 2\int_{\mathbb{R}^d} g(I(gF) + \nabla_v \cdot (EgF))\,\mathrm{d}v \\ &= 2\int_{\mathbb{R}^d} (I(g)g - gE\cdot\nabla_v g)F\,\mathrm{d}v \\ &= \int_{\mathbb{R}^d} (I(|g|^2) - E\cdot\nabla_v|g|^2)F\,\mathrm{d}v - \int_{\mathbb{R}^d} \mathfrak{D}_2(g)F\,\mathrm{d}v \\ &= \int_{\mathbb{R}^d} |g|^2(I(F) + \nabla_v\cdot(EF))\,\mathrm{d}v - \int_{\mathbb{R}^d} \mathfrak{D}_2(g)F\,\mathrm{d}v \\ &= -\int_{\mathbb{R}^d} \mathfrak{D}_2(g)F\,\mathrm{d}v. \end{aligned}$$

We then deduce the result by the following corollary of [70, Theorem 4.3]

$$\int_{\mathbb{R}^d} |f - \mathsf{\Pi} f|^2 \langle v\rangle^{\gamma-\mathfrak{a}} F\,\mathrm{d}v \leq \mathcal{C}_{P,\mathfrak{a},F} \left\|\mathfrak{D}_2(f)\right\|^2_{L^2(F\,\mathrm{d}v)}.$$

□

Part III

Behaviour of Macroscopic models

Chapter 6

Fractional Keller-Segel Equation: Global Well-posedness and Finite Time Blow-up

Abstract

This chapter studies the aggregation diffusion equation

$$\partial_t \rho = \Delta^{\frac{\alpha}{2}} \rho + \lambda \operatorname{div}((\nabla K * \rho)\rho),$$

where $\Delta^{\frac{\alpha}{2}}$ denotes the fractional Laplacian and $\nabla K = \frac{x}{|x|^{\mathrm{a}}}$ is an attractive kernel. This equation is a generalization of the classical Keller-Segel equation, which arises in the modeling of the motion of cells.

In the *diffusion dominated* case $\mathrm{a} < \alpha$, we prove global well-posedness for an L^1_k initial condition with small mass, and in the *fair competition* case $\mathrm{a} = \alpha$ for an $L^1_k \cap L \ln L$ initial condition. In the *aggregation dominated* case $\mathrm{a} > \alpha$, we prove global or local well-posedness for an L^p initial condition, depending on some smallness condition on the L^p norm of the initial condition.

In this last case, we also prove that finite time blow-up of even solutions occurs under some initial mass concentration criteria.

Résumé

Ce chapitre étudie l'équation d'agrégation diffusion suivante

$$\partial_t \rho = \Delta^{\frac{\alpha}{2}} \rho + \lambda \operatorname{div}((\nabla K * \rho)\rho),$$

où $\Delta^{\frac{\alpha}{2}}$ est le Laplacien fractionnaire et $\nabla K = \frac{x}{|x|^{\mathrm{a}}}$ est une force d'interaction attractive. Cette équation est une généralisation de l'équation de Keller-Segel classique qui apparaît dans l'étude du mouvement de bactéries.

Dans le cas à *diffusion dominante* $\mathrm{a} < \alpha$, on démontre l'existence et l'unicité globale en temps pour une condition initiale L^1_k et dans le *cas critique* $\mathrm{a} = \alpha$ pour une condition initiale $L^1_k \cap L \ln L$ avec petite masse. Dans le cas de l'*agrégation dominante* $\mathrm{a} > \alpha$, on démontre l'existence et l'unicité globalement ou localement en temps pour une condition initiale dans L^p. Le caractère global uniquement lorsque la norme L^p est suffisamment petite initialement.

Dans ce dernier cas, on montre aussi qu'il y a un effondrement en temps fini si la masse est initialement suffisamment concentrée.

Contents

6.1 Introduction

The models arising in the context of the chemotaxis of cells have been thoroughly studied in recent years. Among those, the (parabolic-elliptic) Keller-Segel equation models the competition between the aggregation and diffusion of cells (see [40] and references therein for a proper biological and mathematical introduction on the topic). In this chapter we consider a variant of this classical model where the diffusion is modeled with a fractional Laplacian. Such a choice is biologically motivated (see for instance [90, 46] and references therein). From a mathematical point of view, it is then interesting to study how such a diffusion competes with an aggregation field which singularity is up to the Newtonian one.

More precisely for some $(\alpha, \mathrm{a}) \in \mathbb{R}_+^2$, we consider the fractional Keller-Segel equation

$$\partial_t \rho = \Delta^{\frac{\alpha}{2}} \rho + \lambda \operatorname{div}((\nabla K * \rho)\rho), \tag{FKS}$$

where $\lambda > 0$ is a parameter encoding the chemosensitivity, or the intensity of the aggregation. The interaction kernel is given by

$$\nabla K(x) := \frac{x}{|x|^{\mathrm{a}}},$$

and $I := \Delta^{\frac{\alpha}{2}}$ denotes the fractional Laplacian defined by

$$I(u) = \Delta^{\frac{\alpha}{2}} u := c_{d,\alpha} \int_{\mathbb{R}^d} \frac{u(y) - u(x)}{|x - y|^{d+\alpha}} \, \mathrm{d}y. \tag{6.1}$$

The constant $c_{d,\alpha}$ can be written $c_{d,\alpha} = -(2\pi)^\alpha \frac{\omega_{-\alpha}}{\omega_{d+\alpha}} > 0$ where $\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the size of the unit sphere in $\mathbb{R}^d$ when $d \in \mathbb{N}^*$.

Particular cases of equation (FKS) have been studied by numerous authors recently. The classical case corresponds to the choice $\alpha = \mathrm{a} = d = 2$ and has been thoroughly studied in the past years. In [40], the authors show the global well-posedness when the initial mass M_0 is smaller than the critical one $M_c = \frac{4}{\lambda}$. Above this mass, a finite time blow-up is shown to appear. In [61] is also established the well posedness for an L^∞ initial condition. This assumption is sufficient to enjoy the Log-Lipschitz regularity of the nonlinear drift $\nabla K * \rho$, as in this case K is the Newtonian kernel (see for instance [157]). It is possible to relax this assumption to $L \ln L$ initial data [86] or even measure initial data [21]. Large time behaviour is also studied in [40, 56, 86]. In higher dimension, the variant case $\alpha = 2$, $\mathrm{a} = d = 3$ is studied in [71], where a finite time blow-up is obtained under a concentration of initial mass condition.

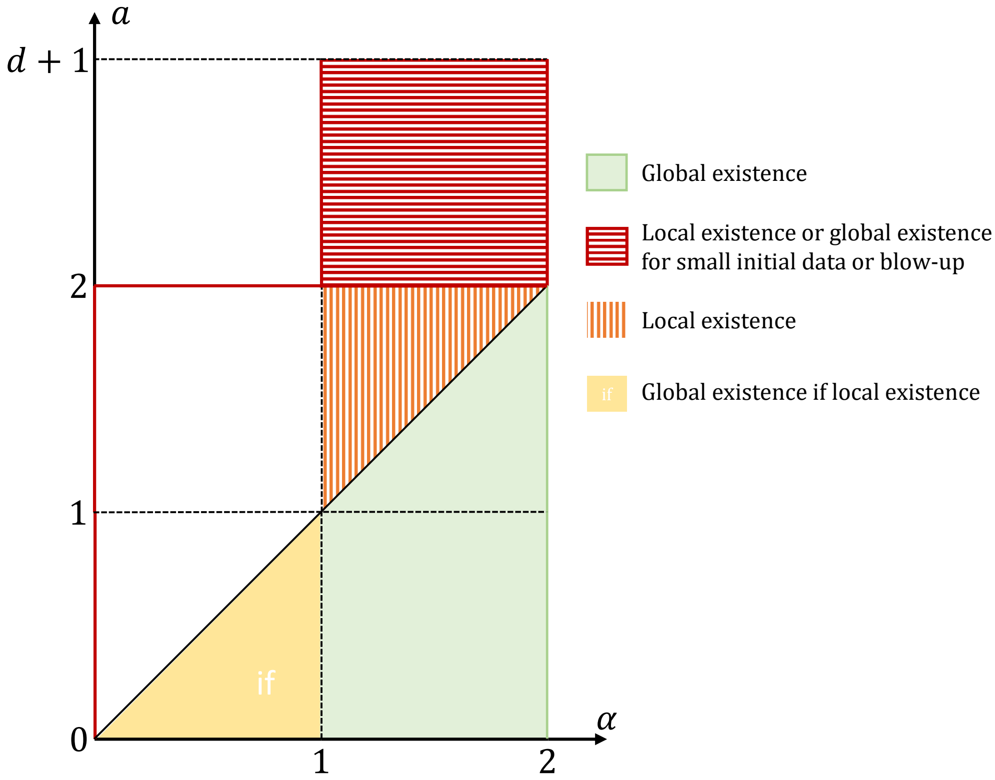

Figure 6.1: Existing results for (FKS) equation.

The literature on the fractional case $\alpha < 2$, is also large and growing and previously known results are summarized in Figure 6.1. In a significant part of it, the kernel K is the Newtonian one ($\mathrm{a} = d$). In the one dimensional case, [46] provides a well posedness result for an L^p initial condition with $p > \frac{1}{\alpha}$ when $\alpha \in (0,1)$ and $p > 1$ when $\mathrm{a} \in (0,1)$, as well as a finite time blow-up of even solutions under some concentration of initial mass criteria. In the case $d \geq 2$, [34] also provides some concentration of initial mass criteria leading to a blow-up of even solutions when $\alpha \in [1,2)$ and non even when $\alpha \in (0,2)$. Still in the Newtonian case, [143] provides similar results in the range $\alpha \in (0,2)$. See also on the limiting case $\alpha = 0$, [28] for $\mathrm{a} \in [0,1)$, [29] for $\mathrm{a} = 1$, and [146] for $\mathrm{a} \in (1,2)$. For $\alpha = 2$ and $\mathrm{a} \in (0,2)$, see [129] and [102], and for $\mathrm{a} = 1$ and $\alpha \in (0,1)$, see [144, 145, 32]. For a wider class of parameters, see also [189] of the second author and [36, 32].

6.2 Main Results

We summarize our results in the following Figure 6.2.

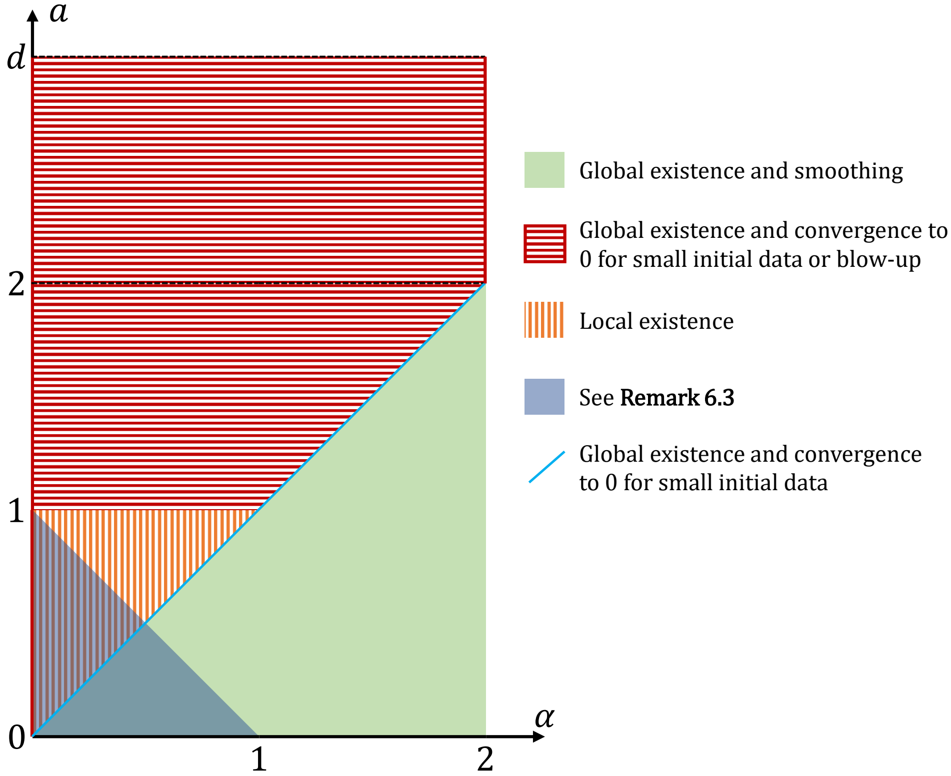

Figure 6.2: Range of application of Theorems 6.2 and 6.7. We emphasize that for $d > 2$ the results extend to the segment $(\alpha, \mathrm{a}) \in \{2\} \times (0, d)$.

We will work on weighted spaces defined by

$$\mathcal{M}_k := \{\rho \in \mathcal{M}, \langle x \rangle^k \rho \in \mathcal{M}\}$$
$$L^p_k := \{\rho \in L^p, \langle x \rangle^k \rho \in L^p\},$$

where $\langle x \rangle = \sqrt{1 + |x|^2}$, $L^p = L^p(\mathbb{R}^d)$ and $\mathcal{M} = \mathcal{M}(\mathbb{R}^d)$ denote the space of bounded measures. We also define the space of functions with finite entropy by

$$L \ln L := \{\rho \in L^1, \rho \ln(\rho) \in L^1\}. \tag{6.2}$$

For $s \in (0,1)$, we will denote by $\mathcal{C}^{\mathrm{S}}_{d,s}$ the best Sobolev's constant such that for any $f \in H^s$

$$\mathcal{C}^{\mathrm{S}}_{d,s} \|f\|^2_{L^{\frac{2d}{d-2s}}} \le |f|^2_{H^s},$$

and for $\mathrm{a} \in (0, d)$ and $p, q > 1$ satisfying $2 - \frac{\mathrm{a}}{d} = \frac{1}{p} + \frac{1}{q}$, we will denote by $\mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},p}$ the best Hardy-Littlewood-Sobolev's constant such that for any $f \in L^p$, $g \in L^q$,

$$\left| \iint_{\mathbb{R}^{2d}} |x-y|^{-\mathrm{a}} f(x) g(y) \,\mathrm{d}x\,\mathrm{d}y \right| \le \mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},p} \|f\|_{L^p} \|g\|_{L^q}. \tag{6.3}$$

Finally for $s \in [0, d)$ and $r = \frac{2d}{d-s}$, we denote $\mathcal{C}_{d,s}^{\mathrm{GNS}}$ the best Gagliardo-Nirenberg-Sobolev's constant such that it holds

$$\mathcal{C}_{d,s}^{\mathrm{GNS}} \|f\|_{L^r}^2 \leq \|f\|_{L^2} |f|_{H^s}.$$

For a given given couple (a, α) we define the following exponents for the L^p spaces which will characterize the integrability of the density

$$p_{\mathrm{a},\alpha} := \frac{d}{d + \alpha - \mathrm{a}} \tag{6.4}$$

$$p_{\mathrm{a}} := p_{\mathrm{a},0} = \frac{d}{d - \mathrm{a}}. \tag{6.5}$$

Taking $\nabla K = \frac{x}{|x|^{\mathrm{a}}}$ let appear two main difficulties. The first one is the singularity at $x = 0$ and the second is the behavior when $x \to \infty$. We will therefore write

$$\nabla K = \mathcal{K}_0 + \mathcal{K}_c = \chi \nabla K + (1 - \chi)\, \nabla K,$$

where $\chi \in C_c^\infty$ verifies $\mathbb{1}_{B_1} \leq \chi \leq \mathbb{1}_{B_2}$. Several parts of our analysis could be easily generalized to more general kernels with similar behavior.

Definition 6.1. *For any $T > 0$, we say that ρ is a weak solution to the* (FKS) *equation on $(0, T)$ with initial condition $\rho^{\mathrm{in}} \in \mathcal{M}$ if it satisfies*

$$\begin{aligned} &\rho \in C^0\left([0,T), \mathcal{M}_{(1-\mathrm{a})_+}\right) && \text{if } \mathrm{a} \in (0,2] \\ &\rho \in C^0\left([0,T), \mathcal{M}\right) \cap L^1_{\mathrm{loc}}\left((0,T), L^{p_{\mathrm{a},2}}\right) && \text{if } \mathrm{a} \in (2, d+2), \end{aligned}$$

and for any $\varphi \in C_c^2$

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\rho(t) - \rho^{\mathrm{in}}\right) \varphi = \int_0^t \int_{\mathbb{R}^d} \rho(s) \left(I(\varphi) - \mathcal{K}_c * (\rho(s) \cdot \nabla\varphi)\right) \\ + \iint_{\mathbb{R}^{2d}} \mathcal{K}_0(x-y) \cdot \left(\nabla\varphi(x) - \nabla\varphi(y)\right) \rho(s, \mathrm{d}x) \rho(s, \mathrm{d}y)\, \mathrm{d}s. \end{aligned} \tag{6.6}$$

We say that this solution is global if we can take $T = +\infty$.

The definition makes sense since it is easy to notice that

$$\begin{aligned} &\mathcal{K}_c * (\rho \nabla\varphi) \in C^0 \cap L^\infty(\langle x \rangle^{\mathrm{a}-1}) \\ &\mathcal{K}_0(x-y)\left(\nabla\varphi(x) - \nabla\varphi(y)\right) \in C^0 \cap L^\infty(\mathbb{R}^{2d}) && \text{if } \mathrm{a} \in (0,2). \end{aligned}$$

Moreover, if $\mathrm{a} \in (2, d+2)$, the last term in Definition 6.1 is bounded thanks to Hardy-Littlewood-Sobolev inequality. Remark that at least formally, this equation conserves the total mass which we will denote by

$$M_0 := \int_{\mathbb{R}^d} \rho^{\mathrm{in}}.$$

First we obtain a global or local well-posedness result, depending on the regime, given in the

Theorem 6.2. *Let* $(\alpha, \mathrm{a}) \in [0,2) \times [0,d)$ *be such that* $\mathrm{a} + \alpha > 1$ *and* $k \in [(1-\mathrm{a})_+, \alpha)$.
- *When* $\mathrm{a} < \alpha$ *and* $\rho^{\mathrm{in}} \in L^1_k$, *there exists a unique and global weak solution to the* (FKS) *equation.*
- *When* $\mathrm{a} = \alpha$, *if* $\rho^{\mathrm{in}} \in L^1_k \cap L \ln L$ *satisfies*

$$\lambda M_0 < C_{\mathrm{a},d} = \frac{4(2\pi)^{\mathrm{a}}}{(d-\mathrm{a})} \left(\frac{\omega_{2d}}{\omega_d}\right) \frac{\omega_{d-\mathrm{a}}}{\omega_{2d-\mathrm{a}}} \max\left(\frac{\omega_{d-\mathrm{a}}}{\omega_{d+\mathrm{a}}}, \frac{\omega^2_{d-\mathrm{a}/2}}{\omega^2_{d+\mathrm{a}/2}}\right), \tag{6.7}$$

then there exists a unique and global weak solution to the (FKS) *equation.*
- *When* $\mathrm{a} > \alpha$ *and* $\rho^{\mathrm{in}} \in L^1_k \cap L^p$ *with* $p \in (p_{\mathrm{a},\alpha}, p_{\mathrm{a}})$, *there exists a time* $T > 0$ *such that there is a unique solution to the* (FKS) *equation on* $(0,T)$. *Moreover there is a constant* $C_{\lambda,p}(M_0)$ *such that if*

$$\|\rho^{\mathrm{in}}\|_{L^p} \le C_{\lambda,p}(M_0), \tag{6.8}$$

then the solution is global.

Remark 6.3. *The constraint* $\mathrm{a} + \alpha > 1$ *comes from the necessity to propagate moments, which is necessary for our notion of solution and gives us compactness. Remark that it is only due to the behavior at infinity of the interaction kernel, which we denoted by* $\mathcal{K}_c$, *and not to the singularity. Therefore, our theorem would hold also for example for the following kernel*

$$\nabla K(x) = \frac{x}{|x|^{\mathrm{a}}}\chi(x) + \frac{x}{|x|^{\gamma}}(1-\chi(x)),$$

for any $\gamma > 1-\alpha$ *and which relaxes the condition* $\mathrm{a}+\alpha > 1$. *It is interesting also to notice that formula* (6.21) *could also provide an alternative definition of solution which does not need moments. However, it is not clear whether it is sufficient to provide compactness.*

Remark 6.4. *The explicit values of* $\mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},p}$ *for* $\mathrm{a} \in (0,d)$ *and* $p = q$ *in* (6.3) *and* $\mathcal{C}^{\mathrm{S}}_{d,s}$ *for* $s \in (0,1)$ *are known, see for instance [147, 148]. Remarking that the HLS conjugate as defined in* (6.3) *of* $p_{\mathrm{a}/2}$ *is itself, it holds*

$$\mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},p_{\mathrm{a}/2}} = \pi^{\frac{\mathrm{a}}{2}} \frac{\Gamma\left(\frac{d-\mathrm{a}}{2}\right)}{\Gamma\left(d-\frac{\mathrm{a}}{2}\right)} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(d)}\right)^{-1+\frac{\mathrm{a}}{d}} = \frac{\omega_{2d-\mathrm{a}}}{\omega_{d-\mathrm{a}}} \left(\frac{\omega_{2d}}{\omega_d}\right)^{\frac{\mathrm{a}-d}{d}}$$

$$\mathcal{C}^{\mathrm{S}}_{d,s} = \frac{2^{2s}\pi^s\,\Gamma\left(\frac{d+2s}{2}\right)}{\Gamma\left(\frac{d-2s}{2}\right)} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(d)}\right)^{\frac{2s}{d}} = (2\pi)^{2s}\frac{\omega_{d-2s}}{\omega_{d+2s}} \left(\frac{\omega_{2d}}{\omega_d}\right)^{\frac{2s}{d}},$$

where we recall that $\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

In the case $\mathrm{a} \le \alpha$, this theorem enlarges the existing result by Biler et al. [36], where global existence is proved for $d = 2, 3$ in the case $\alpha \le \frac{d}{2}$, and is a novelty in higher dimension. Also it is provided with larger class of initial condition, and a uniqueness result. Note that the case $\alpha = \mathrm{a}$ is only the object of some remark in [36, Remark 3.2]. As for the case $\alpha < \mathrm{a} < 2$, it seems it has not been treated yet to the best of the authors' knowledge. See also [34] and [143] for the case $\mathrm{a} = 2$.

Let us briefly sketch the proof of this theorem in the case of an $L \ln L$ initial condition. Formally differentiating the Boltzmann's entropy along (FKS) (see for instance [40, Section 2.2]) provides a control of the $L^1([0,T), L^p)$ for $p \in [1, p_\alpha]$ by fractional Sobolev's embedding, for any initial mass in the *diffusion dominated* case and for small initial mass in the *fair competition* case. Then a slight modification of standard coupling argument enables to obtain stability in this space when $p \in [1, p_{\mathrm{a}})$ and uniqueness when $p = p_{\mathrm{a}}$. The other assumption on the initial condition are meant to control the $L^1([0,T), L^{p_{\mathrm{a}}})$ norm of the solution in the different regimes.

When global existence holds, we also retrieve some additional properties as a quantitative rate of convergence to 0 in the *aggregation dominated* case and a gain of local integrability in the *diffusion dominated* case.

Theorem 6.5. *Let $(\alpha, \mathrm{a}) \in [0,2) \times [0,d)$ and ρ be a solution of the (FKS) equation as given by Theorem 6.2.*
- *When $\mathrm{a} < \alpha$, the gain of integrability is given for any $p \in (1, p_{\mathrm{a}})$ by*

$$\|\rho\|_{L^p} \le C M_0\, t^{-\frac{d}{\alpha q}} + C_\lambda(M_0).$$

- *When $\alpha < \mathrm{a}$ and for a given $p \in (p_{\mathrm{a},\alpha}, p_{\mathrm{a}})$, $\|\rho^{\mathrm{in}}\|_{L^p} < C_{\lambda,p}(M_0)$ defined by (6.8), then there exists a constant $C = C_{\mathrm{a},\alpha,p}(\rho^{\mathrm{in}}) > 0$ such that*

$$\|\rho\|_{L^p} \le C M_0\, t^{-\frac{d}{\alpha q}}.$$

- *When $\mathrm{a} = \alpha$, the condition becomes*

$$\lambda M_0 < C_{\mathrm{a},d,p} = \frac{4\mathcal{C}^{\mathrm{S}}_{d,\mathrm{a}/2}}{p(d-\mathrm{a})\mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},r}}, \quad \frac{1}{r} := \frac{p}{p+1}\frac{1}{p} + \frac{1}{p+1}\frac{1}{p_{\mathrm{a}}},$$

which gives both a gain of integrability and an asymptotic behavior for any $p \in (1, p_{\mathrm{a}})$

$$\lambda M_0 \le C_{\mathrm{a},d,p} \implies \|\rho\|_{L^p} \le C M_0\, t^{-\frac{d}{\alpha q}}, \tag{6.9}$$

where C depends only on M_0, d, p, a and α.

Remark 6.6. *If ρ is a weak solution to the (FKS) equation as given by definition 6.1 with $\mathrm{a} = \alpha$ and $\lambda M_0 < C_{d,\mathrm{a},p}$ for a given $p > 1$, we are not able to assert the uniqueness unless we assume that $\rho^{\mathrm{in}} \in L \ln L$.*

Finally we obtain a finite time blow-up for even solutions to (FKS) under some concentration of mass condition stated in the

Theorem 6.7. *Let $(\alpha, \mathrm{a}) \in [0,2) \times [1,d)$ be such that $\alpha < \mathrm{a}$, $k \in (0,\alpha)$ and $\rho \in C^0(\mathbb{R}_+, L^1_k)$ be an even nonnegative weak solution to the (FKS) equation with initial condition $\rho^{\mathrm{in}} \in L^1_k$ verifying*

$$\int_{\mathbb{R}^d} \rho^{\mathrm{in}}(x)\, \langle x \rangle^k \,\mathrm{d}x \le C^* \lambda^{\frac{k}{2(\mathrm{a}-k)}} M_0^{\frac{2\mathrm{a}-k}{2(\mathrm{a}-k)}} \qquad \text{if } \alpha > 1 \tag{6.10}$$

$$\int_{\mathbb{R}^d} \rho^{\mathrm{in}}(x)\, |x|^k \,\mathrm{d}x \le C_2^* M_0 \quad \text{and} \quad \lambda M_0 \ge C_3^* \qquad \text{if } \alpha < 1 \tag{6.11}$$

for given constants C^*, C_2^*, C_3^* depending only on d, a, α and k. Then the solution ceases to exist in finite time.

The proof of this theorem relies on the time differentiation of an adequate moment, which is adapted to the fractional diffusion and not Newtonian aggregation case, and which leads to a contradiction.

One of the strength of the result of Theorem 6.7, even if it deals only with even solutions, is that it applies to weakly singular interactions, i.e. a < 2. Indeed it seems that so far most of finite time blow-up results for aggregation fractional diffusion equation dealt with the case of a Newtonian interaction at the exception of [32, Theorem 2.2], which deals with interactions of the from $\frac{x}{|x|}$ near the origin. Considering a less singular kernel than the Newtonian erases some algebraic facilities and requires a thinner estimation of the competing terms. We emphasize that it also covers the purely aggregative case $\alpha = 0$, giving stronger results than [29, 146] for the case a ≥ 2. For a ≤ 2, the blow-up was already proved in [28] using a Lagrangian point of view.

Finally, let us comment about the disjunction of the different global existence and finite time blow-up conditions. Condition (6.8) in Theorem 6.2 is heuristically in contradiction with the assumption of Theorem 6.7. First remark that if we require that ρ^{in} is concentrated around zero, for instance with a condition of the type $\|\rho^{\mathrm{in}}\|_{L^1_k} < CM_0$ for a given constant C which does not depend on ρ^{in}, then the condition of blow-up (6.10) is equivalent to

$$\lambda M_0 \geq C',$$

where C' is a positive constant that depends only on a, α, k and d. Moreover, in a more general setting, for $k > 0$, $q = p' \in (1, \infty)$ and $\rho \in L^1_k \cap L^p$, the following inequality

$$\int_{\mathbb{R}^d} \rho \leq C \left(\int_{\mathbb{R}^d} \rho \left\langle x \right\rangle^k \right)^{\frac{d}{d+kq}} \|\rho\|_{L^p}^{\frac{kq}{d+kq}},$$

holds with C depending only on d, k and q. With fixed M_0, this inequality is enough to exclude a priori (6.8) from (6.10) or (6.11), at least in the range of arbitrarily large (or small) $\|\rho^{\mathrm{in}}\|_{L^p}$ or $\int_{\mathbb{R}^d} \rho^{\mathrm{in}} \left\langle x \right\rangle^k$. When this is not the case, we expect that no other behavior appear in the remaining cases.

We restrict ourselves to check that in the simple case $\alpha = \mathrm{a} = 2 < d$, the global well-posedness condition (6.7) is coherent with the classical large mass blow-up criteria. Indeed take a solution to (FKS) in that case, it is possible to consider initial condition $\rho^{\mathrm{in}} \in L^1_2$ and then classically

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \rho\, |x|^2 &= \int_{\mathbb{R}^d} \rho\, \Delta(|x|^2) - \lambda \iint_{\mathbb{R}^{2d}} \nabla K(x-y) \cdot (x-y)\, \rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &= 2dM_0 - \lambda M_0^2 \\ &= 2dM_0 \left(1 - \frac{\lambda M_0}{2d}\right),\end{aligned}$$

so that the condition $\lambda M_0 > 2d$ yields to final time blow-up. And since $\omega_{a+2} = \frac{2\pi}{a}\omega_a$, it

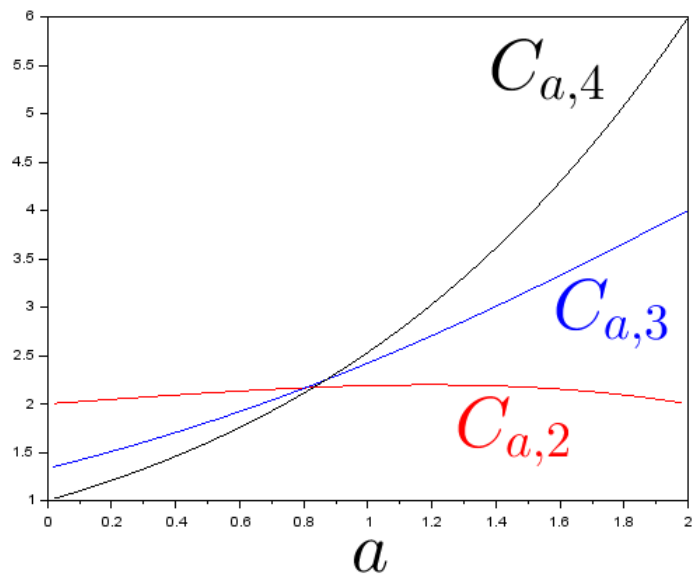

Figure 6.3: Lower bound of the threshold of condition (6.7) for $d = 2, 3, 4$ and $\mathrm{a} \in (0, 2)$. For the case $\mathrm{a} \leq \frac{1}{2}$ see Remark 6.3.

holds

$$\begin{aligned} C_{2,d} &= \frac{4(2\pi)^2}{(d-2)} \frac{\omega_{2d}}{\omega_d} \frac{\omega_{d-2}}{\omega_{2d-2}} \max\left(\frac{\omega_{d-2}}{\omega_{d+2}}, \frac{\omega_{d-1}^2}{\omega_{d+1}^2}\right) \\ &= \frac{4(2\pi)^2}{(d-2)} \frac{d-2}{2d-2} \max\left(\frac{d(d-2)}{(2\pi)^2}, \frac{(d-1)^2}{(2\pi)^2}\right) \\ &= 2(d-1) < 2d, \end{aligned}$$

so that the two conditions cannot be realized simultaneously.

6.3 Proof of Theorem 6.2 and Theorem 6.5

6.3.1 A Priori estimates.

We begin this section with an a priori moment estimate given in the

Proposition 6.1 (Propagation of weight)**.** *Assume $1 - \mathrm{a} < \alpha$ and $\mathrm{a} < 2$ if $\alpha < 1$ and let $k \in [(1-\mathrm{a})_+, \alpha)$ and ρ be a solution of the* (FKS) *equation with initial condition $\rho^{\mathrm{in}} \in L^1_k$. Then*

$$\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^1_k).$$

Proof. Let $m = \langle x\rangle^k$ and $M_k = \|\rho\|_{L^1_k}$. When $k \geq 1$, the convexity of m leads to

$$\begin{aligned}\frac{\mathrm{d}M_k}{\mathrm{d}t} &= \int_{\mathbb{R}^d} \rho\, I(m) - \lambda \iint_{\mathbb{R}^{2d}} h_m(x,y)\rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\leq \int_{\mathbb{R}^d} \rho\, I(m),\end{aligned} \tag{6.12}$$

where $h_m(x,y) = \frac{(\nabla m(x)-\nabla m(y))\cdot(x-y)}{2|x-y|^{\mathrm{a}}} \geq 0$. From [31, Remark 4.2] and [132, Proposition 2.2], we know that for any $k \in (0,\alpha)$,

$$I(m) \leq C_{\alpha,k} m(x)\langle x\rangle^{-\alpha}. \tag{6.13}$$

Since $m(x)\langle x\rangle^{-\alpha} \leq 1$, the following inequality holds

$$\frac{\mathrm{d}M_k}{\mathrm{d}t} \leq C_{\alpha,k}M_0.$$

When $k \in [1-\mathrm{a}, \alpha\wedge 1)$, we decompose the second term in (6.12) as the sum of the integral over the domain $|x-y| < R$ and its complementary for a given $R > 0$. Since ∇m is Lipschitz, we obtain

$$\begin{aligned}-\iint_{|x-y|\leq R} h_m(x,y)\rho(\mathrm{d}x)\rho(\mathrm{d}y) &\leq C\iint_{|x-y|\leq R} |x-y|^{2-\mathrm{a}}\rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\leq CR^{2-\mathrm{a}}M_0^2,\end{aligned}$$

where $C = \|\nabla^2 m\|_{L^\infty}$. The other part can be controlled as follows

$$\begin{aligned}-\iint_{|x-y|>R} h_m(x,y)\rho(\mathrm{d}x)\rho(\mathrm{d}y) &\leq k\iint_{|x-y|>R} \frac{(x\cdot y)(\langle x\rangle^{k-2}+\langle y\rangle^{k-2})}{|x-y|^{\mathrm{a}}}\rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\leq 2k\iint_{|x-y|>R} \frac{(x\cdot y)\langle x\rangle^{k-2}}{|x-y|^{\mathrm{a}}}\rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\leq 4k\iint_{|x-y|>R,|x|>|y|} \frac{|x||y|\langle x\rangle^{k-2}}{|x-y|^{\mathrm{a}}}\rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\leq 4k\,(I_1+I_2),\end{aligned}$$

where

$$\begin{aligned}I_1 &= \iint_{|x-y|>R,2|x|<|y|} \frac{|y|\langle x\rangle^{k-1}}{|x-y|^{\mathrm{a}}}\rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ I_2 &= \iint_{|x-y|>R,|x|<|y|<2|x|} \frac{|y|\langle x\rangle^{k-1}}{|x-y|^{\mathrm{a}}}\rho(\mathrm{d}x)\rho(\mathrm{d}y).\end{aligned}$$

Since $|x-y| > ||y|-|x|| > |y||1-|x|/|y|| > |y|/2$ when $|y| > 2|x|$, we get

$$\begin{aligned}I_1 &\leq 2^{\mathrm{a}}\iint_{|x-y|>R,2|x|<|y|} |y|^{1-\mathrm{a}}\langle x\rangle^{k-1}\rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\leq 2^{\mathrm{a}}M_{1-\mathrm{a}}M_0.\end{aligned}$$

For I_2, we use the fact that $|y| < 2\langle x \rangle$ to obtain

$$\begin{aligned} I_2 &\leq \frac{2}{R^{\mathrm{a}}} \iint_{|x-y|>R, |x|<|y|<2|x|} \langle x \rangle^k \rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\leq \frac{2}{R^{\mathrm{a}}} M_k M_0. \end{aligned}$$

Combining these three inequalities with (6.12) and (6.13), we obtain

$$\frac{\mathrm{d}M_k}{\mathrm{d}t} \leq C_{\alpha,k} M_{k-\alpha} + \lambda M_0 \left(2^{1-\mathrm{a}} M_{1-\mathrm{a}} + \frac{2}{R^{\mathrm{a}}} M_k + C R^{2-\mathrm{a}} M_0 \right).$$

In particular, since $1 - \mathrm{a} \leq k$ and $k - \alpha < 0$, we get

$$\frac{\mathrm{d}M_k}{\mathrm{d}t} \leq M_0 \left(C_{\alpha,k,M_0} + \lambda \left(2^{1-\mathrm{a}} + \frac{2}{R^{\mathrm{a}}} \right) M_k \right).$$

By Gronwall's Lemma, this leads to

$$M_k \leq \left(M_k^{\mathrm{in}} + \frac{C_{\alpha,k,M_0}}{\lambda\, C_{\mathrm{a},R}} \right) e^{\lambda\, C_{\mathrm{a},R} M_0 t},$$

which proves the result. □

The second type of estimates are a priori bounds of integrability. Let us first briefly emphasize that the quantities we estimate will take the form

$$\int_{\mathbb{R}^d} \Phi(u(x))\, \mathrm{d}x,$$

where $u \geq 0$ and $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing convex mapping such that $\Phi(0) = 0$ and $u \mapsto u\, \Phi''(u) \in L^1_{\mathrm{loc}}$. Then we can define

$$\Psi(u) := \int_0^u v\, \Phi''(v)\, \mathrm{d}v \tag{6.14}$$

$$\psi(u) := \frac{1}{2} \int_0^u \sqrt{\Phi''}. \tag{6.15}$$

For $p = q' > 1$ and $u \geq 0$, we recover Lebesgue norms and Boltzmann's entropy as follow

$$\begin{aligned} \Phi_p(u) := \tfrac{1}{p-1} u^p &\implies \Psi_p(u) = u^p \\ & \psi_p(u) = \tfrac{2}{\sqrt{p}} u^{p/2} \\ \Phi_1(u) := u \ln(u) &\implies \Psi(u) = u \\ & \psi_1(u) = 2u^{1/2}. \end{aligned}$$

Lemma 6.2 (General estimate)**.** *Assume that* $(\alpha, \mathrm{a}) \in (0, 2] \times (0, d)$ *(with* $\alpha \neq 2$ *if* $d = 2$*) and let* ρ *be a smooth solution to the* (FKS) *equation,* Φ *be a nondecreasing convex* *mapping,* Ψ *and* ψ *be defined respectively by* (6.14) *and* (6.15) *and* $b \in (1, p_{\mathrm{a}})$*. Then there holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d} \Phi(\rho)\right) \leq \lambda(d-\mathrm{a})\mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},b}\|\rho\|_{L^s}\|\Psi(\rho)\|_{L^b} - |\psi(\rho)|^2_{H^{\frac{\alpha}{2}}}, \tag{6.16}$$

$$\leq \lambda(d-\mathrm{a})\mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},b}\|\rho\|_{L^s}\|\Psi(\rho)\|_{L^b} - \mathcal{C}^{\mathrm{S}}_{d,\alpha/2}\|\psi(\rho)\|^2_{L^{\tilde{b}}}, \tag{6.17}$$

where

$$\frac{1}{s} = 2 - \frac{\mathrm{a}}{d} - \frac{1}{b}, \quad \frac{2}{\tilde{b}} = 1 - \frac{\alpha}{d}.$$

Proof. We define the "Carré du Champs" and the Φ-dissipation by

$$\Gamma(u,v) := \frac{c_{d,\alpha}}{2}\int_{\mathbb{R}^d} \frac{(u(y)-u(x))(v(y)-v(x))}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x \tag{6.18}$$

$$\mathfrak{D}_\Phi(u) := \Gamma(u, \Phi'(u)), \tag{6.19}$$

where $c_{d,\alpha}$ is defined in (6.1). With these definitions, we have

$$\int_{\mathbb{R}^d} I(u)v = \int_{\mathbb{R}^d} u\, I(v) = -\int_{\mathbb{R}^d} \Gamma(u,v).$$

In particular, since Φ is convex,

$$\int_{\mathbb{R}^d} I(u)\Phi'(u) = -\int_{\mathbb{R}^d} \mathfrak{D}_\Phi(u) \leq 0.$$

We remark that

$$\begin{aligned}|\psi(u)-\psi(v)|^2 &= \left|\int_u^v \sqrt{\Phi''}\right|^2 \\ &\leq \left(\int_u^v \mathrm{d}s\right)\left(\int_u^v \Phi''\right) \\ &\leq (u-v)(\Phi'(u)-\Phi'(v)),\end{aligned}$$

which by definition (6.18) leads to

$$\Gamma(\psi(u),\psi(u)) \leq \Gamma(u,\Phi'(u)).$$

Therefore

$$|\psi(u)|^2_{H^{\frac{\alpha}{2}}} = \int_{\mathbb{R}^d} \Gamma(\psi(u),\psi(u)) \leq \int_{\mathbb{R}^d} \Gamma(u,\Phi'(u)) = \int_{\mathbb{R}^d} \mathfrak{D}_\Phi(u). \tag{6.20}$$

Let ρ be a nonnegative solution to the (FKS) equation. Then formally

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d} \Phi(\rho)\right) &= \int_{\mathbb{R}^d} \Phi'(\rho)\, I(\rho) - \lambda\Phi''(\rho)\nabla\rho\cdot(\nabla K * \rho)\rho \\ &= -\int_{\mathbb{R}^d} \mathfrak{D}_\Phi(\rho) - \int_{\mathbb{R}^d} \lambda\nabla(\Psi(\rho))\cdot(\nabla K*\rho) \\ &= -\int_{\mathbb{R}^d} \mathfrak{D}_\Phi(\rho) + \lambda\int_{\mathbb{R}^d} \Psi(\rho)(\Delta K * \rho) \\ &= -\int_{\mathbb{R}^d} \mathfrak{D}_\Phi(\rho) + \lambda(d-\mathrm{a})\int_{\mathbb{R}^d}\left(\frac{1}{|x|^{\mathrm{a}}} * \rho\right).\end{aligned} \tag{6.21}$$

We remark that by Hardy-Littlewood-Sobolev inequality, we have

$$(d-\mathrm{a})\int_{\mathbb{R}^d}\left(\frac{1}{|x|^{\mathrm{a}}}*\rho\right)\Psi(\rho)\leq(d-\mathrm{a})\,\mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},b}\,\|\rho\|_{L^s}\,\|\Psi(\rho)\|_{L^b}\,,$$

and by formula (6.20) and Sobolev's embedding, we have

$$-\int_{\mathbb{R}^d}\mathfrak{D}_\Phi(\rho)\leq-|\psi(\rho)|^2_{H^{\frac{\alpha}{2}}}\leq-\mathcal{C}^{\mathrm{S}}_{d,\alpha/2}\|\psi(\rho)\|^2_{L^{\tilde b}},$$

which ends the proof. □

Proposition 6.3 ($L\ln L$ estimate)**.** *Let $\mathrm{a}=\alpha$ and ρ be a smooth function satisfying the (FKS) equation with initial condition $\rho^{\mathrm{in}}\in L\ln L$. Then it holds*

$$\int_{\mathbb{R}^d}\rho\ln(\rho)+4C^{-1}_{\mathrm{a},d}\left(\lambda M_0-C_{\mathrm{a},d}\right)\int_0^t|\sqrt{\rho}|^2_{H^{\frac{\mathrm{a}}{2}}}\leq\int_{\mathbb{R}^d}\rho^{\mathrm{in}}\ln(\rho^{\mathrm{in}}),$$

with $\rho=\rho(t,\cdot)$ and

$$C_{\mathrm{a},d}=\frac{4(\mathcal{C}^{\mathrm{GNS}}_{d,\mathrm{a}/2})^2}{(d-\mathrm{a})\mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},p_{\mathrm{a}/2}}}.$$

Moreover if $\lambda M_0<C_{\mathrm{a},d}$ and for some $T,k>0$, $\rho\in L^\infty((0,T),L^1_k)$, then

$$\rho\in L^1((0,T),L^{p_{\mathrm{a}}}).\tag{6.22}$$

Remark 6.8. *The explicit values for $\mathcal{C}^{\mathrm{GNS}}_{d,\mathrm{a}/2}$ do not seem to be known, however the following lower bound holds*

$$\mathcal{C}^{\mathrm{GNS}}_{d,s}\geq\max\left(\mathcal{C}^{\mathrm{S}}_{d,s/2},(\mathcal{C}^{\mathrm{S}}_{d,s})^{1/2}\right).\tag{6.23}$$

Indeed, one way to get the Gagliardo-Nirenberg-Sobolev inequality is to first use Sobolev's inequality and then interpolation between H^s spaces

$$\mathcal{C}^{\mathrm{S}}_{d,s/2}\|f\|^2_{L^r}\leq|f|^2_{H^{\frac{s}{2}}}\leq\|f\|_{L^2}|f|_{H^s}.$$

A second way is to first interpolate between Lebesgue spaces and then to use Sobolev's inequality

$$(\mathcal{C}^{\mathrm{S}}_{d,s})^{1/2}\|f\|^2_{L^r}\leq(\mathcal{C}^{\mathrm{S}}_{d,s})^{1/2}\|f\|_{L^2}\|f\|_{L^{r_2}}\leq\|f\|_{L^2}\|f\|_{H^s},$$

where $r_2:=\frac{2d}{d-2s}$.

Proof. We use inequality (6.16) for $\Phi=\Phi_1$, $\psi_1(u)=2u^{1/2}$ and $b=s=p_{\mathrm{a}/2}$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho\,\ln(\rho)\leq\lambda\,(d-\mathrm{a})\,\mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},b}\,\|\rho\|^2_{L^b}-|\psi_1(\rho)|^2_{H^{\frac{\mathrm{a}}{2}}}\,.$$

Then, by Gagliardo-Nirenberg-Sobolev's inequality, we have

$$\begin{aligned}\left(\mathcal{C}^{\mathrm{GNS}}_{d,\mathrm{a}/2}\right)^2\|\rho\|^2_{L^b}&=\left(\mathcal{C}^{\mathrm{GNS}}_{d,\mathrm{a}/2}\right)^2\|\rho^{1/2}\|^4_{L^{2b}}\\&\leq\|\rho^{1/2}\|^2_{L^2}|\rho^{1/2}|^2_{H^{\frac{\mathrm{a}}{2}}}=M_0|\rho^{1/2}|^2_{H^{\frac{\mathrm{a}}{2}}}.\end{aligned}$$

Hence, since $\psi_1(u) = 2u^{1/2}$, we have

$$4\left(\mathcal{C}^{\mathrm{GNS}}_{d,\mathrm{a}/2}\right)^2 \|\rho\|^2_{L^b} \le M_0 |\psi_1(\rho)|^2_{H^{\frac{\mathrm{a}}{2}}}.$$

This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \rho\, \ln(\rho) \le C^{-1}_{\mathrm{a},d}\left(\lambda M_0 - C_{\mathrm{a},d}\right)|\psi_1(\rho)|^2_{H^{\frac{\mathrm{a}}{2}}},$$

which proves the first assertion. Formula (6.22) comes form the fact for $k > 0$, defining $m(x) := \langle x\rangle^k$ and $\lambda_k > 0$ such that $\int_{\mathbb{R}^d} e^{-\lambda_k \langle x\rangle^k}\mathrm{d}x = 1$, with $h(u) = u\ln u - u + 1 \ge 0$ it holds

$$\int_{\mathbb{R}^d} \frac{\rho}{M_0}\ln\frac{\rho}{M_0} = \int_{\mathbb{R}^d} h\left(\frac{\rho}{M_0}e^{\lambda_k m}\right)e^{-\lambda_k m} + \int_{\mathbb{R}^d}\frac{\rho}{M_0}\ln(e^{-\lambda_k m}) \ge -\lambda_k\int_{\mathbb{R}^d}\frac{\rho}{M_0}m,$$

and then

$$\int_{\mathbb{R}^d}\rho\ln\rho \ge M_0\ln M_0 - \lambda_k\int_{\mathbb{R}^d}\rho\, m.$$

Combined with the following Sobolev's inequality

$$4\mathcal{C}^{\mathrm{S}}_{d,\mathrm{a}/2}\|\rho\|_{L^{p_\mathrm{a}}} = \mathcal{C}^{\mathrm{S}}_{d,\mathrm{a}/2}\|\psi_1(\rho)\|^2_{L^{2p_\mathrm{a}}} \le |\psi_1(\rho)|^2_{H^{\frac{\mathrm{a}}{2}}},$$

it yields

$$\begin{aligned} 0 \le \int_{\mathbb{R}^d}(\rho\ln(\rho) + \lambda_k\langle x\rangle^k\rho) - M_0\ln M_0 + 4\mathcal{C}^{\mathrm{S}}_{d,\mathrm{a}/2}C^{-1}_{\mathrm{a},d}\left(C_{\mathrm{a},d} - \lambda M_0\right)\int_0^t \|\rho\|_{L^{p_\mathrm{a}}} \\ \le \int_{\mathbb{R}^d}\rho^{\mathrm{in}}\ln(\rho^{\mathrm{in}}) + \lambda_k\|\rho\|_{L^\infty(0,T;L^1_k)} - M_0\ln M_0, \end{aligned}$$

and the conclusion follows. □

Proposition 6.4 (L^p estimates)**.** *Let $(\alpha,\mathrm{a}) \in [0,2)\times[0,d)$. Then, when $\mathrm{a} < \alpha$ and $p = q' \in (1, p_\mathrm{a})$, we get a gain of integrability from L^1 to L^p and a global in time propagation of the L^p norm*

$$\|\rho(t)\|_{L^p} \le CM_0\max\left(t^{-\frac{d}{\alpha q}}, M_0^{\frac{d}{q(\alpha-\mathrm{a})}}\right), \tag{6.24}$$

where $C > 0$ is a constant depending on d, a, α, p and λ. When $\mathrm{a} > \alpha$, then for any $p \in (p_{\mathrm{a},\alpha}, p_\mathrm{a})$, there exists two constants $C = C_{\mathrm{a},\alpha,p} > 0$ and $C^{\mathrm{in}} = C_{\mathrm{a},\alpha,p}(\|\rho^{\mathrm{in}}\|_{L^p})$ such that

$$\|\rho^{\mathrm{in}}\|_{L^p} < CM_0(\lambda M_0)^{-\frac{d}{(\mathrm{a}-\alpha)q}} \implies \|\rho\|_{L^p} \le C^{\mathrm{in}}M_0 t^{-\frac{d}{\alpha q}} \tag{6.25}$$

$$\|\rho^{\mathrm{in}}\|_{L^p} > CM_0(\lambda M_0)^{-\frac{d}{(\mathrm{a}-\alpha)q}} \implies \rho \in L^\infty((0,T), L^p) \tag{6.26}$$

$$\|\rho^{\mathrm{in}}\|_{L^p} = CM_0(\lambda M_0)^{-\frac{d}{(\mathrm{a}-\alpha)q}} \implies \rho \in L^\infty(\mathbb{R}_+, L^p), \tag{6.27}$$

where $T < C_{\mathrm{a},\alpha,p}(\lambda, M_0)\|\rho^{\mathrm{in}}\|^{-pb}_{L^p}$ with

$$b = \frac{\alpha}{p(\alpha-\mathrm{a}) + d(p-1)}.$$

When $\mathrm{a} = \alpha$, *then there exists a constant*

$$C_{\mathrm{a},d,p} = \frac{4\mathcal{C}^{\mathrm{S}}_{d,\alpha/2}}{(d-\mathrm{a})\mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},r}},$$

such that for any $p \in (1, p_{\mathrm{a}})$,

$$\lambda M_0 \leq C_{\mathrm{a},d,p} \implies \|\rho\|_{L^p} \leq M_0 (C^{\mathrm{in}} b)^{-\frac{1}{b}} t^{-\frac{d}{\alpha q}} \tag{6.28}$$

$$\lambda M_0 \geq C_{\mathrm{a},d,p} \implies \rho \in L^\infty((0,T), L^p), \tag{6.29}$$

where C^{in} *is a nonnegative constant depending on the initial data and*

$$T > \frac{1}{bC^{\mathrm{in}}} \left(\frac{M_0}{\|\rho^{\mathrm{in}}\|_{L^p}} \right)^{\frac{\alpha q}{d}}.$$

Remark 6.9. *The critical mass is clearly not optimal since we could use optimal constants in the Gagliardo-Nirenberg type embedding, as it is done in the* $L \ln L$ *estimate, instead of using Sobolev's embedding and interpolation between Lebesgue spaces.*

Proof. We will separate the proof into several steps.

Step 1. Differential inequality for the L^p ***norm.*** We recall that

$$\frac{1}{r} = \frac{p}{p+1}\frac{1}{p} + \frac{1}{p+1}\frac{1}{p_{\mathrm{a}}}.$$

Since $p < p_{\mathrm{a}}$, it implies that $r \in (p, p_{\mathrm{a}})$ and in particular $r/p > 1$. Therefore, by taking $\Phi = \Phi_p$, $r = s$ and $b = r/p$ in inequality (6.17) and defining $\tilde{r} = \frac{p\tilde{b}}{2}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{\mathbb{R}^d} \Phi_p(\rho) \right) \leq \lambda \mathcal{C}_{\mathrm{a},r} \|\rho\|^{p+1}_{L^r} - \tfrac{\mathcal{C}_\alpha}{p} \|\rho\|^p_{L^{\tilde{r}}}, \tag{6.30}$$

where $\mathcal{C}_{\mathrm{a},r} = (d-\mathrm{a})\mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},r}$, $\mathcal{C}_\alpha = 4\mathcal{C}^{\mathrm{S}}_{d,\alpha/2}$ and

$$\frac{p+1}{r} = 2 - \frac{\mathrm{a}}{d} \tag{6.31}$$

$$\frac{p}{\tilde{r}} = 1 - \frac{\alpha}{d}. \tag{6.32}$$

We also remark that

$$\begin{aligned} r \leq \tilde{r} &\Leftrightarrow \tfrac{1}{p}\left(1 - \tfrac{\alpha}{d}\right) \leq \tfrac{1}{p+1}\left(2 - \tfrac{\mathrm{a}}{d}\right) \\ &\Leftrightarrow \left(1 + \tfrac{1}{p}\right)(d-\alpha) \leq (2d - \mathrm{a}) \\ &\Leftrightarrow p \geq \frac{d-\alpha}{d+\alpha-\mathrm{a}}. \end{aligned}$$

Since $p \geq p_{\mathrm{a},\alpha} \geq \frac{d-\alpha}{d+\alpha-\mathrm{a}}$, we deduce that $r \leq \tilde{r}$.

We will now use interpolation between Lebesgue spaces to express the left hand side of (6.30) in terms of M_0 and the L^p norm only. Let $\varepsilon \in (0,1)$ to be chosen later and

$$b_0 := \frac{\mathrm{a} - \alpha(1-\varepsilon)}{\varepsilon d(p-1)} = \frac{\alpha}{d(p-1)} + \frac{\mathrm{a}-\alpha}{\varepsilon d(p-1)} \tag{6.33}$$

$$\theta_1 := \frac{\varepsilon p}{p+1}(1+b_0) \tag{6.34}$$

$$\theta_2 := \frac{(1-\varepsilon)p}{p+1} \tag{6.35}$$

$$\theta_0 := 1 - \theta_1 - \theta_2. \tag{6.36}$$

Since $p > 1$ and $\varepsilon \in (0,1)$, we deduce that $\theta_2 \in [0,1)$. Moreover, using the respective definitions (6.31) and (6.32) of r and $\tilde{r}$, we have

$$\begin{aligned} \frac{\theta_1}{p} + \frac{\theta_2}{\tilde{r}} + \theta_0 &= \frac{\varepsilon(1+b_0)p}{p+1}\left(\frac{1}{p} - 1\right) + \frac{(1-\varepsilon)p}{p+1}\left(\frac{1}{\tilde{r}} - 1\right) + 1 \\ &= \frac{1}{p+1}\left(\varepsilon(1-p)(1+b_0) + (1-\varepsilon)\left(1 - \frac{\alpha}{d} - p\right) + p + 1\right) \\ &= \frac{1}{p+1}\left(\varepsilon(1-p) - \frac{\mathrm{a} - \alpha(1-\varepsilon)}{d} + 2 - \frac{\alpha}{d} - \varepsilon(1-p) + \varepsilon\frac{\alpha}{d}\right) \\ &= \frac{1}{p+1}\left(2 - \frac{\mathrm{a}}{d}\right) = \frac{1}{r}. \end{aligned}$$

Therefore, if we can choose $\varepsilon \in (0,1)$ such that $(\theta_0, \theta_1) \in [0,1]^2$, we obtain by interpolation

$$\|\rho\|_{L^r}^{p+1} \le M_0^{\theta_0(p+1)} \|\rho\|_{L^p}^{p\varepsilon(1+b_0)} \|\rho\|_{L^{\tilde{r}}}^{p(1-\varepsilon)} = A^\varepsilon B^{1-\varepsilon}.$$

Then, by using the standard Young inequality $a^\varepsilon b^{1-\varepsilon} \le \varepsilon a + (1-\varepsilon)b$, for any $\varepsilon_0 > 0$, we have

$$A^\varepsilon B^{1-\varepsilon} = \left(\left(\frac{1-\varepsilon}{\varepsilon_0}\right)^{\frac{1-\varepsilon}{\varepsilon}} A\right)^\varepsilon \left(\frac{\varepsilon_0 B}{1-\varepsilon}\right)^{1-\varepsilon} \le C_{\varepsilon,\varepsilon_0} A + \varepsilon_0 B,$$

with $C_{\varepsilon,\varepsilon_0} = \varepsilon\left(\frac{1-\varepsilon}{\varepsilon_0}\right)^{\frac{1-\varepsilon}{\varepsilon}}$. Coming back to (6.30), it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d} \Phi_p(\rho)\right) \le (\lambda \mathcal{C}_{\mathrm{a},r})^{1/\varepsilon} C_{\varepsilon,\varepsilon_0} M_0^{\theta_0(p+1)/\varepsilon} \|\rho\|_{L^p}^{p(1+b_0)} + \left(\varepsilon_0 - \tfrac{\mathcal{C}_\alpha}{p}\right) \|\rho\|_{L^{\tilde{r}}}^p, \tag{6.37}$$

where we take ε_0 smaller than $\mathcal{C}_\alpha/p$. Since $1 \le p \le \tilde{r}$, again by interpolation, we get

$$\|\rho\|_{L^p}^{p(1+b_1)} \le M_0^{pb_1} \|\rho\|_{L^{\tilde{r}}}^p,$$

with

$$b_1 = \frac{\alpha}{d(p-1)}.$$

Thus, inequality (6.37) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^p}^p \le C_1 M_0^{\theta_0(p+1)/\varepsilon} \|\rho\|_{L^p}^{p(1+b_0)} - C_2 M_0^{-pb_1} \|\rho\|_{L^p}^{p(1+b_1)}, \tag{6.38}$$

where $C_1 = (p-1)(\lambda \mathcal{C}_{\mathrm{a},r})^{1/\varepsilon} C_{\varepsilon,\varepsilon_0}$ and $C_2 = (p-1)\left(\frac{\mathcal{C}_\alpha}{p} - \varepsilon_0\right)$.

Step 2. Conditions on ε. We still have to verify that we can choose ε so that $(\theta_0, \theta_1) \in [0,1]^2$. By definition (6.34) of θ_1, we get

$$\begin{aligned}\theta_1 \geq 0 &\Leftrightarrow b_0 \geq -1 \\ &\Leftrightarrow \mathrm{a} - \alpha + \alpha\varepsilon > -\varepsilon d(p-1) \\ &\Leftrightarrow \varepsilon \geq \frac{\alpha - \mathrm{a}}{\alpha + d(p-1)} = \varepsilon_m.\end{aligned}$$

Moreover, by definition (6.36) of θ_0

$$\begin{aligned}\theta_0 \geq 0 &\Leftrightarrow \theta_1 + \theta_2 \leq 1 \\ &\Leftrightarrow \frac{p}{p+1}(1 + \varepsilon b_0) \leq 1 \\ &\Leftrightarrow \varepsilon b_0 \leq \frac{1}{p} \\ &\Leftrightarrow \frac{\mathrm{a} - \alpha(1-\varepsilon)}{d(p-1)} \leq \frac{1}{p} \\ &\Leftrightarrow \varepsilon \leq 1 - \frac{1}{\alpha}\left(\mathrm{a} - \frac{d}{q}\right) = \varepsilon_M.\end{aligned}$$

Since $p < p_{\mathrm{a}}$, $\varepsilon_M < 1$. Let us check that it is nonnegative. We have

$$\varepsilon_M \geq 0 \Leftrightarrow \mathrm{a} - \frac{d}{q} \leq \alpha \Leftrightarrow \frac{1}{q} \geq \frac{\mathrm{a} - \alpha}{d}.$$

Since $q = p' \geq 1$, this is always verified when $\mathrm{a} \leq \alpha$. When $\mathrm{a} > \alpha$, it is verified by hypothesis since we can also read previous formula as

$$\varepsilon_M \geq 0 \Leftrightarrow p \geq \frac{d}{d + \alpha - \mathrm{a}} = p_{\mathrm{a},\alpha}.$$

When $\mathrm{a} < \alpha$, we also have to verify that $\varepsilon_m \leq \varepsilon_M$. We have, indeed

$$\begin{aligned}\frac{\varepsilon_M}{\varepsilon_m} &= \frac{(p(\alpha - \mathrm{a}) + d(p-1))(\alpha + d(p-1))}{p\alpha(\alpha - \mathrm{a})} \\ &= \frac{p\alpha(\alpha - \mathrm{a}) + d(p(p-1)(\alpha - \mathrm{a}) + \alpha(p-1) + d(p-1)^2)}{p\alpha(\alpha - \mathrm{a})} \\ &= 1 + d(p-1)\frac{p(\alpha - \mathrm{a}) + \alpha + d(p-1)}{p\alpha(\alpha - \mathrm{a})} > 1.\end{aligned}$$

Therefore, since $\theta_2 \geq 0$ and $\theta_0 + \theta_1 + \theta_2 = 1$, we proved that for any $\varepsilon \in [\max(\varepsilon_m, 0), \min(\varepsilon_M, 1)]$,

$$(\theta_0, \theta_1, \theta_2) \in [0,1]^3.$$

By looking at (6.38), we want to take ε which minimizes b_0. Hence, we take

$$\begin{aligned}\varepsilon &= \varepsilon_m \text{ when } \mathrm{a} < \alpha, \\ \varepsilon &= \varepsilon_M \text{ when } \mathrm{a} > \alpha.\end{aligned}$$

Step 3. Case $\mathrm{a} < \alpha$**.** In this case, we have $\varepsilon = \varepsilon_m$, which yields $b_0 = -1$. Moreover, since

$$\theta_0(p+1) = p + 1 - (1-\varepsilon)p = 1 + \varepsilon p,$$

by (6.38), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^p}^p \leq C_1 M_0^{p+1/\varepsilon} - C_2 M_0^{-pb_1}\|\rho\|_{L^p}^{p(1+b_1)}.$$

Then, either

$$C_2 M_0^{-pb_1}\|\rho\|_{L^p}^{p(1+b_1)} \leq 2C_1 M_0^{p+1/\varepsilon}, \tag{6.39}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^p}^p \leq -\tfrac{1}{2}C_2 M_0^{-pb_1}\|\rho\|_{L^p}^{p(1+b_1)}. \tag{6.40}$$

Inequality (6.39) is equivalent to

$$\|\rho\|_{L^p}^p \leq \left(\frac{2C_1}{C_2}\right)^{\frac{1}{1+b_1}} M_0^{p+\frac{1}{\varepsilon(1+b_1)}} =: C(M_0),$$

and by Gronwall's inequality, (6.40) leads to

$$\|\rho\|_{L^p}^p \leq \left(\tfrac{1}{2}C_2 M_0^{-pb_1} b_1 t\right)^{-1/b_1} = M_0^p\left(\tfrac{b_1}{2}C_2 t\right)^{-1/b_1}$$

Step 4. Case $\mathrm{a} > \alpha$**.** In this case, we have

$$\varepsilon = \varepsilon_M = \frac{p(\alpha - \mathrm{a}) + d(p-1)}{\alpha p} = \frac{1}{pb},$$

which by definition (6.33) leads to

$$\begin{aligned} b_0 &= \frac{1}{d(p-1)}\left(\alpha + \frac{\mathrm{a}-\alpha}{\varepsilon}\right) \\ &= \frac{\alpha}{d(p-1)}\left(\frac{p(\alpha-\mathrm{a}) + d(p-1) + p(\mathrm{a}-\alpha)}{p(\alpha-\mathrm{a}) + d(p-1)}\right) \\ &= \frac{\alpha}{p(\alpha-\mathrm{a}) + d(p-1)} = b, \end{aligned}$$

and by inequality (6.38), to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^p}^p \leq C_1 M_0^{\theta_0(p+1)/\varepsilon}\|\rho\|_{L^p}^{p(1+b)} - C_2 M_0^{-pb_1}\|\rho\|_{L^p}^{p(1+b_1)}.$$

As remarked previously, $\varepsilon = \varepsilon_M \geq 0$. Therefore, since $\mathrm{a} > \alpha$,

$$b = b_1 + \frac{\mathrm{a}-\alpha}{\varepsilon d(p-1)} \geq b_1. \tag{6.41}$$

The estimate on the L^p norm is then obtained by analyzing the corresponding ODE which is of the form

$$y'(t) = Ay(t)^{1+b} - By(t)^{1+b_1},$$

with A and B nonnegative. It has a fixed point at $y = 0$ and at

$$y^\sharp = \left(\frac{B}{A}\right)^{\frac{1}{b-b_1}} \geq 0.$$

Therefore, when $y(0) \in [0, y^\sharp)$, it yields $y(t) \in [0, y^\sharp)$ for any $t > 0$, and since $y' \leq 0$ in this interval, it implies the existence of a constant $C = C(y(0)^{\text{in}}) < 1$ such that

$$Ay^{1+b} \leq CBy^{1+b_1}.$$

It implies that

$$y' \leq -(1-C)By^{1+b_1},$$

which, by Gronwall's inequality, leads to

$$\forall t \in \mathbb{R}_+, y \leq \frac{1}{(y(0)^{-b_1} + b_1(1-C)Bt)^{\frac{1}{b_1}}} \leq \frac{M_0^p}{(b_1(1-C)C_2 t)^{\frac{1}{b_1}}}.$$

When $y(0) > y^\sharp$, we can still write that

$$y' \leq Ay^{1+b}.$$

It implies that the solution is bounded in $[0, T]$ for some $T > 0$ and

$$\forall t \in [0,T], y(t) \leq \frac{1}{(y(0)^{-b} - bAt)^{\frac{1}{b}}}$$
$$T < \frac{1}{bAy(0)^b}.$$

We deduce the corresponding results for the L^p norm of ρ by Gronwall's inequality. When $y = y^\sharp$, all we get that y is constant and therefore that $\|\rho\|_{L^p}^p \leq y^\sharp$ for any $t > 0$. We can compute more precisely

$$y^\sharp = \left(\frac{C_2 M_0^{-pb_1}}{C_1 M_0^{\theta_0(p+1)/\varepsilon}}\right)^{\frac{1}{b-b_1}} = \left(\frac{C_2}{C_1}\right)^{\frac{1}{b-b_1}} \left(M_0^{-\theta_0(p+1)/\varepsilon - pb_1}\right)^{\frac{1}{b-b_1}}.$$

Now by the definitions of C_1 and C_2 in step 1, by (6.41) and the definition (6.36) of θ_0, we have

$$\theta_0(p+1) = 1 - \varepsilon pb = 0$$
$$(b - b_1)\varepsilon = \frac{\mathrm{a} - \alpha}{d(p-1)}$$
$$C_1 = (p-1)(\lambda \mathcal{C}_{\mathrm{a},r})^{1/\varepsilon} C_{\varepsilon,\varepsilon_0}$$
$$C_2 = (p-1)\left(\tfrac{\mathcal{C}_\alpha}{p} - \varepsilon_0\right).$$

This leads to

$$y^\sharp = \left(\frac{\mathcal{C}_\alpha - \varepsilon_0 p}{(\lambda \mathcal{C}_{\mathrm{a},r})^{1/\varepsilon} C_{\varepsilon,\varepsilon_0} p}\right)^{\frac{1}{b-b_1}} \left(M_0^{p(b-b_1)-1/\varepsilon}\right)^{\frac{1}{b-b_1}}$$
$$= C_{\mathrm{a},\alpha,p}^p M_0^{p - \frac{d(p-1)}{\mathrm{a}-\alpha}} \lambda^{\frac{d(p-1)}{\mathrm{a}-\alpha}}.$$

Step 5. Case $\mathrm{a} = \alpha$. When $\mathrm{a} = \alpha$, by definition (6.33), b_0 does not depend on ε and

$$b = b_0 = b_1 = \frac{\alpha}{d(p-1)}$$
$$\theta_0(p+1) = 1 - \varepsilon p b.$$

Moreover, we can take any $\varepsilon \in (\varepsilon_m, \varepsilon_M] = (0, d/(\alpha q)]$. Thus, by inequality (6.38), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^p}^p &\le C_1 M_0^{(1-\varepsilon pb)/\varepsilon}\|\rho\|_{L^p}^{p(1+b)} - C_2 M_0^{-pb}\|\rho\|_{L^p}^{p(1+b)} \\ &\le \|\rho\|_{L^p}^{p(1+b)} M_0^{-pb}\left(C_1 M_0^{1/\varepsilon} - C_2\right).\end{aligned}$$

The left hand side will be negative when

$$M_0 \le \left(\frac{C_2}{C_1}\right)^{\varepsilon} = \frac{(\mathcal{C}_\alpha/p - \varepsilon_0)^{\varepsilon}}{\lambda\, \mathcal{C}_{\mathrm{a},r}}\left(\frac{\varepsilon_0}{1-\varepsilon}\right)^{1-\varepsilon}\varepsilon^{-\varepsilon} = u_\varepsilon(\varepsilon_0). \tag{6.42}$$

Taking ε_0 maximizing the right hand side, we get

$$\varepsilon_0 = (1-\varepsilon)\mathcal{C}_\alpha/p$$
$$u_\varepsilon(\varepsilon_0) = \frac{\mathcal{C}_\alpha}{p\lambda\mathcal{C}_{\mathrm{a},r}} = \frac{C_{\mathrm{a},d,p}}{\lambda}.$$

When this is the case, then $C^{\mathrm{in}} := |C_1 M_0^{1/\varepsilon} - C_2| > 0$ and by Gronwall's inequality

$$\forall t \in \mathbb{R}_+, \|\rho\|_{L^p}^p \le \frac{1}{(\|\rho^{\mathrm{in}}\|_{L^p}^{-pb} + bM_0^{-pb}C^{\mathrm{in}}t)^{\frac{1}{b}}} \le \frac{M_0^p}{(bC^{\mathrm{in}}t)^{\frac{1}{b}}},$$

which proves (6.28). When $M_0 > M_0^*$ we only get the existence of $T > 0$ such that

$$\forall t \in [0,T], \|\rho\|_{L^p}^p \le \frac{1}{(\|\rho^{\mathrm{in}}\|_{L^p}^{-pb} - bM_0^{-pb}C^{\mathrm{in}}t)^{\frac{1}{b}}}.$$

Moreover, T verifies

$$T > \frac{1}{bC^{\mathrm{in}}}\left(\frac{M_0}{\|\rho^{\mathrm{in}}\|_{L^p}}\right)^{pb},$$

which proves (6.29). □

Corollary 6.5. *When* $\mathrm{a} < \alpha$ *and* $\rho^{\mathrm{in}} \in L^1$, *then for any* $p < p_\alpha$

$$\rho \in L^1_{\mathrm{loc}}(\mathbb{R}_+, L^p), \tag{6.43}$$

which holds in particular if $p = p_{\mathrm{a}}$. *When* $\mathrm{a} > \alpha$ *and* $\rho^{\mathrm{in}} \in L^p$ *for a given* $p \in (p_{\mathrm{a},\alpha}, p_{\mathrm{a}})$, *then there exists* $T > 0$ *such that*

$$\rho \in L^p((0,T), L^{\tilde{r}}), \tag{6.44}$$

where $\tilde{r} = pp_\alpha \ge p_{\mathrm{a}}$. *Moreover, if* (6.25) *is verified,*

$$\rho \in L^p_{\mathrm{loc}}(\mathbb{R}_+, L^{\tilde{r}}).$$

Proof. Equation (6.43) comes from inequality (6.24) by remarking that $p < p_\alpha$ implies $d/(\alpha q) \leq 1$ and integrating in time. Equation (6.44) is a consequence of (6.37), which by integrating in time leads to

$$\|\rho\|_{L^p}^p(t) + C_2 \int_0^t \|\rho(s)\|_{L^{\tilde{r}}}^p \,\mathrm{d}s \leq \|\rho^{\mathrm{in}}\|_{L^p}^p + C_1 M_0^{\theta_0(p+1)/\varepsilon} \int_0^t \|\rho(s)\|_{L^p}^{p(1+b_0)} \,\mathrm{d}s.$$

If $\rho \in L^\infty([0,T], L^p)$, then we deduce that

$$C_2 \int_0^t \|\rho(s)\|_{L^{\tilde{r}}}^p \,\mathrm{d}s \leq \|\rho^{\mathrm{in}}\|_{L^p}^p + C_1 M_0^{\theta_0(p+1)/\varepsilon} T \|\rho\|_{L^\infty([0,T],L^p)}^{p(1+b_0)},$$

and we conclude by using (6.25) or (6.26). □

6.3.2 Tightness and coupling.

For the rest of the section we consider some given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The expectation with respect to $\mathbb{P}$ will be denoted $\mathbb{E}$. We first provide a generalization of [60, Proposition 3.1] in the

Lemma 6.6. *Let be* $\mathrm{a} \leq d$, $k \geq 1$ *and* $p \geq p_{\mathrm{a}}$. *There exists a constant* C *depending only on* d, p, a *such that for any* $\rho_1, \rho_2 \in \mathcal{P}_k \cap L^p$ *and* $X, \bar{X}$ *two i.i.d. random variables of law* ρ_1 *(respectively* $Y, \bar{Y}$ *two i.i.d. of law* ρ_2*), it holds when* $p > p_{\mathrm{a}}$

$$\mathbb{E}\left[|X-Y|^{k-1}\left|\nabla K(X-\bar{X}) - \nabla K(Y-\bar{Y})\right|\right] \leq C C_{\rho_1,\rho_2} \mathcal{E}_k, \tag{i}$$

and when $p = p_{\mathrm{a}}$,

$$\mathbb{E}\left[|X-Y|^{k-1}\left|\nabla K(X-\bar{X}) - \nabla K(Y-\bar{Y})\right|\right] \leq C C_{\rho_1,\rho_2} \mathcal{E}_k \left(1 + \frac{\ln_-(\mathcal{E}_k)}{k}\right), \tag{ii}$$

where $C_{\rho_1,\rho_2} = 1 + \|\rho_1\|_{L^p} + \|\rho_2\|_{L^p}$ *and* $\mathcal{E}_k = \mathbb{E}\left|[X-Y|^k\right]$.

Remark 6.10. *The point* (i) *of this Lemma has been extensively used in the literature (See for instance [59, 58, 102, 189]). So has the point* (ii) *in the Newtonian case* $\mathrm{a} = d$ *and thus* $p_{\mathrm{a}} = \infty$ *(see for instance [157, 115, 60]). Since we did not found its generalization to a general Riesz interaction kernel* $\mathrm{a} \in (0, d)$, *we provide more detail. A similar technique can be found in [133].*

Proof. We start with the classical inequality (see [115, (3.9)], [102, Lemma 2.5], [58, (3.26)], [59, (3.5)]) which holds for any $(x, y) \in (\mathbb{R}^d)^2$

$$|\nabla K(x) - \nabla K(y)| \leq \left(\left|\nabla^2 K(x)\right| + \left|\nabla^2 K(y)\right|\right)|x-y|.$$

Then denote $\pi = \mathcal{L}(X, Y) = \mathcal{L}(\bar{X}, \bar{Y}) \in \mathcal{P}(\mathbb{R}^{2d})$.

Step 1. Proof of (i)***.*** We assume here that $p > p_{\mathrm{a}}$. Then we have

$$\begin{aligned}
&\mathbb{E}\left[|X-Y|^{k-1}\left|\nabla K(X-\bar{X})-\nabla K(Y-\bar{Y})\right|\right] \\
&\quad \leq \mathbb{E}\left[|X-Y|^{k-1}(|X-Y|+\left|\bar{X}-\bar{Y}\right|)\left(|\nabla^2 K(X-\bar{X})|+|\nabla^2 K(Y-\bar{Y})|\right)\right] \\
&\quad := \mathcal{I}_1+\mathcal{I}_2.
\end{aligned}$$

We first estimate $\mathcal{I}_1$. Since X and $\bar{X}$ are independent we get

$$\begin{aligned}
\mathcal{I}_1 &= \mathbb{E}\left[|X-Y|^k\,\mathbb{E}\left[\left(|\nabla^2 K(X-\bar{X})|+|\nabla^2 K(Y-\bar{Y})|\right)|X,Y\right]\right] \\
&= \mathbb{E}\left[|X-Y|^k\left(\iint_{\mathbb{R}^{2d}}\left(|\nabla^2 K(X-x)|+|\nabla^2 K(Y-y)|\right)\pi(\mathrm{d}x,\mathrm{d}y)\right)\right] \\
&= \mathbb{E}\left[|X-Y|^k\left(\int_{\mathbb{R}^d}|\nabla^2 K(X-x)|\rho_1(x)\,\mathrm{d}x+\int_{\mathbb{R}^d}|\nabla^2 K(Y-y)|\rho_2(y)\,\mathrm{d}y\right)\right].
\end{aligned}$$

But since $|\nabla^2 K| \leq C_{\mathrm{a}}|x|^{-\mathrm{a}}$ with $C_{\mathrm{a}} = \max(1-\mathrm{a},\mathrm{a})$, we obtain

$$\begin{aligned}
C_{\mathrm{a}}^{-1}\int_{\mathbb{R}^d}|\nabla^2 K(X-x)|\rho_1(x)\,\mathrm{d}x &\leq \int_{\mathbb{R}^d}|X-x|^{-\mathrm{a}}\rho_1(x)\,\mathrm{d}x \\
&\leq \int_{|X-x|\leq r}|X-x|^{-\mathrm{a}}\rho_1(x)\,\mathrm{d}x + r^{-\mathrm{a}}\|\rho_1\|_{L^1} \\
&\leq \|\rho_1\|_{L^p}\left(\int_{|x|<r}|x|^{-\mathrm{a}q}\,\mathrm{d}x\right)^{1/q} + r^{-\mathrm{a}}\|\rho_1\|_{L^1},
\end{aligned}$$

where $q = p'$ and $r > 0$. Since $p > p_{\mathrm{a}}$, we get $\mathrm{a}q < d$ so that $|x|^{-\mathrm{a}q}$ is locally integrable and we obtain

$$C_{\mathrm{a}}^{-1}\int_{\mathbb{R}^d}|\nabla^2 K(X-x)|\rho_1(x)\,\mathrm{d}x \leq C_K\, r^{d/q}\|\rho_1\|_{L^p} + r^{-\mathrm{a}}\|\rho_1\|_{L^1},$$

where $C_K = \left(\frac{\omega_d}{d-\mathrm{a}q}\right)^{1/q}$.

Step 2. Proof of (ii)***.*** Note that for any $(x,y) \in (\mathbb{R}^d)^2$ and $r > 0$, it holds

$$|\nabla K(x)-\nabla K(y)| \leq \begin{cases} |\nabla K(x)|+|\nabla K(y)| & \text{if } |x|\wedge|y| \leq r \\ (|\nabla\nabla K(x)|+|\nabla\nabla K(y)|)\,|x-y| & \text{else.} \end{cases}$$

So that

$$\begin{aligned}
&\mathbb{E}\left[|X-Y|^{k-1}\left|\nabla K(X-\bar{X})-\nabla K(Y-\bar{Y})\right|\right] \leq \\
&\quad \mathbb{E}\left[|X-Y|^{k-1}\left(|\nabla K(X-\bar{X})|+|\nabla K(Y-\bar{Y})|\right)\mathbb{1}_{|X-\bar{X}|\wedge|Y-\bar{Y}|\leq r}\right] \\
&\quad + \mathbb{E}[|X-Y|^{k-1}\left(|X-Y|+\left|\bar{X}-\bar{Y}\right|\right) \\
&\qquad\qquad \left(|\nabla^2 K(X-\bar{X})|+|\nabla^2 K(Y-\bar{Y})|\right)\mathbb{1}_{|X-\bar{X}|\wedge|Y-\bar{Y}|>r}] \\
&\quad =: \mathcal{I}_1+\mathcal{I}_2.
\end{aligned}$$

To estimate $\mathcal{I}_1$, we write

$$\begin{aligned}\mathcal{I}_1 &= I_1^1 + I_1^2 + I_1^3 \\ &:= \mathbb{E}\left[|X-Y|^{k-1}\left(|\nabla K(X-\bar{X})| + |\nabla K(Y-\bar{Y})|\right)\mathbb{1}_{|X-\bar{X}|\vee|Y-\bar{Y}|\le r}\right] \\ &\quad + \mathbb{E}\left[|X-Y|^{k-1}\left(|\nabla K(X-\bar{X})| + |\nabla K(Y-\bar{Y})|\right)\mathbb{1}_{|X-\overline{X}|>r\ge|Y-\overline{Y}|}\right] \\ &\quad + \mathbb{E}\left[|X-Y|^{k-1}\left(|\nabla K(X-\bar{X})| + |\nabla K(Y-\bar{Y})|\right)\mathbb{1}_{|X-\overline{X}|\le r<|Y-\overline{Y}|}\right].\end{aligned}$$

Then, for the estimate of I_1^1, we get by independence of $\bar{X}$ and X (respectively $\bar{Y}$ and Y)

$$\begin{aligned}I_1^1 &= \mathbb{E}\left[\mathbb{E}\left[\left(|\nabla K(X-\bar{X})| + |\nabla K(Y-\bar{Y})|\right)\mathbb{1}_{|X-\overline{X}|\vee|Y-\overline{Y}|\le r}|X,Y\right]|X-Y|^{k-1}\right] \\ &\le \mathbb{E}\left[\left(\int_{|X-x|\le r}\frac{\rho_1(x)}{|X-x|^{\mathrm{a}-1}}\,\mathrm{d}x + \int_{|Y-y|\le r}\frac{\rho_2(y)}{|Y-y|^{\mathrm{a}-1}}\,\mathrm{d}y\right)|X-Y|^{k-1}\right] \\ &\le \left(\|\rho_1\|_{L^{p_\mathrm{a}}} + \|\rho_2\|_{L^{p_\mathrm{a}}}\right)\left(\int_{|z|\le r}|z|^{-(\mathrm{a}-1)\frac{d}{\mathrm{a}}}dz\right)^{\frac{\mathrm{a}}{d}}\mathbb{E}\Big[|X-Y|^{k-1}\Big].\end{aligned}$$

Since

$$\int_{|z|\le r}|z|^{-(\mathrm{a}-1)\frac{d}{\mathrm{a}}}dz = \omega_d\int_0^r u^{d-1-(\mathrm{a}-1)\frac{d}{\mathrm{a}}}\,\mathrm{d}s = \frac{\mathrm{a}\omega_d}{d}r^{\frac{d}{\mathrm{a}}} =: (C'_{d,\mathrm{a}}r)^{\frac{d}{\mathrm{a}}},$$

we get

$$I_1^1 \le C'_{d,\mathrm{a}}r\left(\|\rho_1\|_{L^{p_\mathrm{a}}} + \|\rho_2\|_{L^{p_\mathrm{a}}}\right)\mathbb{E}\left[|X-Y|^{k-1}\right].$$

For I_1^2, we have

$$\begin{aligned}I_1^2 &\le \mathbb{E}\left[\frac{2}{|Y-\overline{Y}|^{\mathrm{a}-1}}\mathbb{1}_{|Y-\overline{Y}|\le r}|X-Y|^{k-1}\right] \\ &= 2\,\mathbb{E}\left[\left(\int_{|Y-y|\le r}\frac{\rho_2(y)}{|Y-y|^{\mathrm{a}-1}}\,\mathrm{d}y\right)|X-Y|^{k-1}\right] \\ &\le 2C'_{d,\mathrm{a}}r\,\|\rho_2\|_{L^{p_\mathrm{a}}}\,\mathbb{E}\left[|X-Y|^{k-1}\right].\end{aligned}$$

We then estimate I_1^3 similarly. Combining the above estimates, we obtain

$$\begin{aligned}\mathcal{I}_1 &\le 3C'_{d,\mathrm{a}}r\left(\|\rho_1\|_{L^{p_\mathrm{a}}} + \|\rho_2\|_{L^{p_\mathrm{a}}}\right)\mathbb{E}\left[|X-Y|^{k-1}\right] \\ &\le 3C'_{d,\mathrm{a}}r\left(\|\rho_1\|_{L^{p_\mathrm{a}}} + \|\rho_2\|_{L^{p_\mathrm{a}}}\right)\mathbb{E}\left[|X-Y|^{k}\right]^{(k-1)/k}.\end{aligned}$$

Next, we estimate $\mathcal{I}_2$ by writing

$$\begin{aligned}\mathcal{I}_2 &= C_\mathrm{a}(I_2^1 + I_2^2) \\ &:= C_\mathrm{a}\,\mathbb{E}\left[|X-Y|^k\left(\frac{1}{|X-\overline{X}|^\mathrm{a}} + \frac{1}{|Y-\overline{Y}|^\mathrm{a}}\right)\mathbb{1}_{|X-\overline{X}|\wedge|Y-\overline{Y}|>r}\right] \\ &\quad + C_\mathrm{a}\,\mathbb{E}\left[|X-Y|^{k-1}|\overline{X}-\overline{Y}|\left(\frac{1}{|X-\overline{X}|^\mathrm{a}} + \frac{1}{|Y-\overline{Y}|^\mathrm{a}}\right)\mathbb{1}_{|X-\overline{X}|\wedge|Y-\overline{Y}|>r}\right].\end{aligned}$$

First we easily obtain since $\mathbb{1}_{a\wedge b\geq r} = \mathbb{1}_{a\geq r}\mathbb{1}_{b\geq r}$

$$I_2^1 = \mathbb{E}\left[|X-Y|^k\,\mathbb{E}\left[\left(\frac{1}{|X-\overline{X}|^{\mathrm{a}}} + \frac{1}{|Y-\overline{Y}|^{\mathrm{a}}}\right)\mathbb{1}_{|X-\overline{X}|\wedge|Y-\overline{Y}|>r}|X,Y\right]\right]$$
$$\leq \mathbb{E}\left[|X-Y|^k\left(\int_{|X-x|\geq r}\frac{\rho_1(x)}{|X-x|^{\mathrm{a}}}\,\mathrm{d}x + \int_{|Y-y|\geq r}\frac{\rho_2(y)}{|Y-y|^{\mathrm{a}}}\,\mathrm{d}y\right)\right].$$

We then consider two cases: $r>1$ and $0<r\leq 1$. For $r\leq 1$, we get

$$\int_{|X-x|\geq r}\frac{\rho_1(x)}{|X-x|^{\mathrm{a}}}\,\mathrm{d}x = \int_{|X-x|>1}\frac{\rho_1(x)}{|X-x|^{\mathrm{a}}}\,\mathrm{d}x + \int_{|X-x|\in[r,1]}\frac{\rho_1(x)}{|X-x|^{\mathrm{a}}}\,\mathrm{d}x$$
$$\leq \|\rho_1\|_{L^1} + \|\rho_1\|_{L^{p_{\mathrm{a}}}}\left(\int_{|X-x|\in[r,1]}\frac{1}{|X-x|^d}\,\mathrm{d}x\right)^{\frac{\mathrm{a}}{d}}$$
$$\leq \|\rho_1\|_{L^1} + \omega_d\|\rho_1\|_{L^{p_{\mathrm{a}}}}\ln_-(r)^{\frac{\mathrm{a}}{d}}$$
$$\leq C_d\left(\|\rho_1\|_{L^{p_{\mathrm{a}}}} + \|\rho_1\|_{L^1}\right)(1+\ln_- r).$$

For the case $r>1$, it is clear to obtain

$$\int_{|X-x|\geq r}\frac{\rho_1(x)}{|X-x|^{\mathrm{a}}}\,\mathrm{d}x \leq \|\rho_1\|_{L^1}.$$

This yields

$$I_2^1 \leq C_d\left(\|\rho_1\|_{L^{p_{\mathrm{a}}}} + \|\rho_2\|_{L^{p_{\mathrm{a}}}} + 2\right)\mathbb{E}\left[|X-Y|^k\right](1+\ln_- r).$$

On the other hand, by Hölder's inequality

$$I_2^2 \leq \mathbb{E}\left[|\overline{X}-\overline{Y}|^k\left(\frac{1}{|X-\overline{X}|^{\mathrm{a}}} + \frac{1}{|Y-\overline{Y}|^{\mathrm{a}}}\right)\mathbb{1}_{|X-\overline{X}|\wedge|Y-\overline{Y}|>r}\right]^{1/k}$$
$$\times\,\mathbb{E}\left[|X-Y|^k\left(\frac{1}{|X-\overline{X}|^{\mathrm{a}}} + \frac{1}{|Y-\overline{Y}|^{\mathrm{a}}}\right)\mathbb{1}_{|X-\overline{X}|\wedge|Y-\overline{Y}|>r}\right]^{1-1/k}.$$

The second term of the product is some power of the term I_2^1 which has already been dealt with, and so is the second term by symmetry of the roles of (X,Y) and $(\overline{X},\overline{Y})$. So that

$$\mathcal{I}_2 \leq C_{d,\mathrm{a}}\left(\|\rho_1\|_{L^{p_{\mathrm{a}}}} + \|\rho_2\|_{L^{p_{\mathrm{a}}}} + 2\right)\mathbb{E}\left[|X-Y|^k\right](1+\ln_- r).$$

Putting all these estimates together yields for any $r>0$

$$\mathbb{E}[|X-Y|^{k-1}\left|\nabla K(X-\bar{X}) - \nabla K(Y-\bar{Y})\right|]$$
$$\leq C'_{d,\mathrm{a}}\left(\|\rho_1\|_{L^{p_{\mathrm{a}}}} + \|\rho_2\|_{L^{p_{\mathrm{a}}}}\right) r\,\mathbb{E}\left[|X-Y|^k\right]^{1-1/k}$$
$$+ C_{d,\mathrm{a}}\left(\|\rho_1\|_{L^{p_{\mathrm{a}}}} + \|\rho_2\|_{L^{p_{\mathrm{a}}}} + 2\right)\mathbb{E}\left[|X-Y|^k\right](1+\ln_- r).$$

Choosing $r = \mathbb{E}\left[|X-Y|^k\right]^{1/k}$ yields the desired result. □

Proof of Theorem 6.2. Let ρ^{in} be such as the assumptions of Theorem 6.2. For $\varepsilon > 0$ define

$$\kappa_\varepsilon(x) = \begin{cases} \nabla K(x) & \text{if } |x| \geq \varepsilon \\ \varepsilon^{-\mathrm{a}} x & \text{else,} \end{cases}$$

and consider the following nonlinear PDE with smooth coefficient

$$\partial_t \rho_\varepsilon = I(\rho_\varepsilon) + \lambda \operatorname{div}((\kappa_\varepsilon * \rho_\varepsilon)\rho_\varepsilon), \tag{6.45}$$

with the initial condition $\rho_\varepsilon^{\text{in}} = \rho^{\text{in}}$. Since the kernel κ_ε is $(\varepsilon^{-\mathrm{a}})$-Lipschitz, the difficulty for the well posedness of (6.45) does not come from the quadratic nonlinear term. Existence and uniqueness of solution for this nonlinear problem is straightforward in the case $\mathrm{a} \in (1,2)$. Indeed it is sufficient to apply a standard fixed point technique in $C([0,T],\mathcal{P}_k)$ using Wasserstein metric, since in this case the solution a priori enjoys some $k \in (1,\mathrm{a})$ moment. In the case $\mathrm{a} \in (0,1]$, it is no more possible to use the completeness of $C([0,T],\mathcal{P}_k)$, $k > 1$, and we have to proceed by compactness (see [189, Appendix B]).

Then due to Proposition 6.3 (if $\alpha = \mathrm{a}$), Corollary 6.5 (if $\mathrm{a} \neq \alpha$), and Proposition 6.1, $\rho_\varepsilon \in L^1([0,T],L^p) \cap L^\infty([0,T],L^1_k)$ for some $p \geq p_{\mathrm{a}}$ and $T > 0$ depending or not on ρ^{in}, uniformly with respect to $\varepsilon > 0$.

Step 1. Tightness. Let X_0 be a random variable on $\mathbb{R}^d$ of law $M_0^{-1}\rho^{\text{in}}$ and $(Z_t^\alpha)_{t\geq 0}$ be an α-stable Lévy process independent of X_0. We denote by $(X_t^\varepsilon)_{t\geq 0}$ (respectively $(X_t^{\varepsilon'})_{t\geq 0}$) the solution to the following SDE

$$X_t^\varepsilon = X_0 - \lambda \int_0^t \int_{\mathbb{R}^d} \kappa_\varepsilon(X_s^\varepsilon - x)\rho_\varepsilon(\mathrm{d}x)\,\mathrm{d}s + Z_t^\alpha.$$

Note that $(\mu_\varepsilon(t))_{t\geq 0} := (\mathcal{L}(X_t^\varepsilon))_{t\geq 0}$ solves the linear PDE

$$\partial_t \mu_\varepsilon = I(\mu_\varepsilon) + \lambda \operatorname{div}((\kappa_\varepsilon * \rho_\varepsilon)\mu_\varepsilon),$$

with initial condition $\mu_\varepsilon^{\text{in}} = M_0^{-1}\rho^{\text{in}}$. Therefore $\mathcal{L}(X_t^\varepsilon) = M_0^{-1}\rho_\varepsilon(t)$ by uniqueness of solution to this linear PDE with smooth coefficient.

Assume first $0 < 1-\mathrm{a} < \alpha$. It is direct to obtain in this case for any $\gamma > 1$

$$\begin{aligned} \mathcal{K}_\varepsilon^\gamma &:= \iint_{\mathbb{R}^{2d}} |\kappa_\varepsilon(x-y)|^\gamma \rho_\varepsilon(\mathrm{d}x)\rho_\varepsilon(\mathrm{d}y) \\ &\leq C_{\mathrm{a},\gamma} \iint_{\mathbb{R}^{2d}} (|x-y| \vee \varepsilon)^{\gamma(1-\mathrm{a})}\, \rho_\varepsilon(\mathrm{d}x)\rho_\varepsilon(\mathrm{d}y) \\ &\leq C_{\mathrm{a},\gamma} \int_{\mathbb{R}^d} \left(|x|^{(1-\mathrm{a})\gamma} + \varepsilon^{(1-\mathrm{a})\gamma}\right) \rho_\varepsilon(\mathrm{d}x). \end{aligned}$$

Then choose $\gamma = \frac{k}{1-\mathrm{a}} > 1$ and use the symmetry between x and y to get

$$\begin{aligned} \sup_{0<\varepsilon<1} \int_0^T \mathcal{K}_\varepsilon^\gamma(t)\,\mathrm{d}t &\leq \sup_{0<\varepsilon<1} \int_0^T \iint_{\mathbb{R}^{2d}} C_{\mathrm{a},\gamma} \left(|x|^{(1-\mathrm{a})\gamma} + \varepsilon^{(1-\mathrm{a})\gamma}\right) \rho_\varepsilon(\mathrm{d}x)\rho_\varepsilon(\mathrm{d}y)\,\mathrm{d}t \\ &\leq C_{\mathrm{a},\gamma,T} \left(\sup_{\varepsilon>0} \|\rho_\varepsilon\|_{L^\infty((0,T),L^1_k)} + 1\right) < \infty. \end{aligned}$$

Assume now that $\mathrm{a} > 1$. First note that Hardy-Littlewood-Sobolev inequality yields for any $\varepsilon > 0$ and $\gamma > 1$ to be fixed later

$$\mathcal{K}_\varepsilon^\gamma \leq \iint_{\mathbb{R}^{2d}} |x-y|^{-(\mathrm{a}-1)\gamma} \rho_\varepsilon(\mathrm{d}x)\rho_\varepsilon(\mathrm{d}y) \leq C\|\rho_\varepsilon\|^2_{L^{\frac{d}{d+\gamma(1-\mathrm{a})/2}}}.$$

By interpolation between Lebesgue spaces, if $\gamma < \frac{2(p-1)d}{\mathrm{a}-1}$, then

$$\|\rho_\varepsilon\|_{L^{\frac{d}{d+\gamma(1-\mathrm{a})/2}}} \leq \|\rho_\varepsilon\|^\theta_{L^p}\|\rho_\varepsilon\|^{1-\theta}_{L^1},$$

where $\theta = \gamma\frac{(\mathrm{a}-1)q}{2d}$ with $q = p'$. Therefore

$$\begin{aligned}
\sup_{\varepsilon>0}\int_0^T \mathcal{K}_\varepsilon^\gamma(t)\,\mathrm{d}t &\leq \sup_{\varepsilon>0}\int_0^T \|\rho_\varepsilon\|^2_{L^{\frac{d}{d+\gamma(1-\mathrm{a})/2}}}\,\mathrm{d}t \\
&\leq \sup_{\varepsilon>0}\int_0^T \|\rho_\varepsilon\|^{\gamma\frac{(\mathrm{a}-1)q}{d}}_{L^p}\,\mathrm{d}t < \infty,
\end{aligned}$$

provided that $\gamma \in \left(1, \frac{d}{(\mathrm{a}-1)q}\right)$. Then in both cases, denote the stochastic process

$$J_t^\varepsilon = -\lambda\int_0^t\int_{\mathbb{R}^d}\kappa_\varepsilon(X_s^\varepsilon - x)\rho_\varepsilon(\mathrm{d}x)\,\mathrm{d}s,$$

and observe that for any $0 \leq s < t \leq T$, it holds by Hölder's inequality

$$\begin{aligned}
|J_t^\varepsilon - J_s^\varepsilon| &\leq \left|\int_s^t\int_{\mathbb{R}^d}\kappa_\varepsilon(X_u^\varepsilon - x)\rho_\varepsilon(\mathrm{d}x)\,\mathrm{d}u\right| \\
&\leq \int_s^t\int_{\mathbb{R}^d}|\kappa_\varepsilon(X_u^\varepsilon - x)|\,\rho_\varepsilon(\mathrm{d}x)\,\mathrm{d}u \\
&\leq |t-s|^{1/\gamma'}\int_0^T\left(\int_{\mathbb{R}^d}|\kappa_\varepsilon(X_u^\varepsilon - x)|^\gamma\rho_\varepsilon(\mathrm{d}x)\right)^{1/\gamma}\mathrm{d}u,
\end{aligned}$$

so that by the estimates carried out in the beginning of this step and Jensen's inequality

$$\begin{aligned}
\sup_{0<\varepsilon<1}\mathbb{E}\left[\sup_{0\leq s<t\leq T}\frac{|J_t^\varepsilon - J_s^\varepsilon|}{|t-s|^{1/\gamma'}}\right] &\leq \int_0^T\mathbb{E}\left[\left(\int_{\mathbb{R}^d}|\kappa_\varepsilon(X_u^\varepsilon - x)|^\gamma\rho_\varepsilon(\mathrm{d}x)\right)^{1/\gamma}\right]\mathrm{d}u \\
&\leq \int_0^T\left(\mathbb{E}\left[\int_{\mathbb{R}^d}|\kappa_\varepsilon(X_u^\varepsilon - x)|^\gamma\rho_\varepsilon(\mathrm{d}x)\right]\right)^{1/\gamma}\mathrm{d}u \\
&\leq T^{1/\gamma'}\left(\int_0^T\mathcal{K}_\varepsilon^\gamma(t)\,\mathrm{d}t\right)^{1/\gamma} < \infty.
\end{aligned}$$

We then deduce that the family of law of the processes $(J_t^\varepsilon)_{t\in[0,T]}$ is tight in $\mathcal{P}(C([0,T],\mathbb{R}^d))$. Indeed let us denote

$$\mathcal{K}_R := \left\{f \in C([0,T],\mathbb{R}^d), f(0) = 0, \sup_{0\leq s<t\leq T}\frac{|f(t)-f(s)|}{|t-s|^{1/\gamma'}} \leq R\right\},$$

which is compact due to Ascoli-Arzelà's Theorem. By Markov's inequality we get for any $\varepsilon > 0$

$$\mathbb{P}\left((J_t^\varepsilon)_{0\le t\le T} \notin \mathcal{K}_R\right) = \mathbb{P}\left(\sup_{0\le s<t\le T} \frac{|J_t^\varepsilon - J_s^\varepsilon|}{|t-s|^{1/\gamma'}} > R\right)$$
$$\le R^{-1} \sup_{1>\varepsilon>0} \mathbb{E}\left[\sup_{0\le s<t\le T} \frac{|J_t^\varepsilon - J_s^\varepsilon|}{|t-s|^{1/\gamma'}}\right].$$

Hence the family of laws $\mathcal{L}^\varepsilon = \mathcal{L}\left((X_t^\varepsilon = X_0 + J_t^\varepsilon + Z_t^\alpha)_{0\le t\le T}\right) \in \mathcal{P}(D([0,T],\mathbb{R}^d))$ is tight. Thus, we can find a sequence ε_n going to 0 such that $\mathcal{L}_{\varepsilon_n}$ goes weakly to some $\pi \in \mathcal{P}(D([0,T],\mathbb{R}^d))$. For any $t \in [0,T]$, we define $\mathbf{e}_t : g \in D([0,T],\mathbb{R}^d) \mapsto g(t) \in \mathbb{R}^d$ and $\rho(t) := (\mathbf{e}_t)\#\pi \in \mathcal{P}$ the push-forward of ρ by $\mathbf{e}_t$. Since for any $t \in [0,T]$, $(\mathbf{e}_t)\#\mathcal{L}^\varepsilon = \rho_\varepsilon(t)$, $\rho_{\varepsilon_n}(t)$ goes weakly to $\rho(t)$ in $\mathcal{P}_k$,

Step 2. A priori properties of the limit point. By lower semicontinuity of $\|\cdot\|_{L^p}$ and $\|\cdot\|_{L^1_k}$ with respect to the weak convergence of measures and Fatou's Lemma, it holds $\rho \in L^1([0,T],L^p) \cap L^\infty([0,T],L^1_k)$. We now show that ρ satisfies (6.6). Indeed for $\varphi \in C_c^2$ denote

$$\mathcal{F}(\rho,t) = \int_{\mathbb{R}^d} \left(\rho(t) - \rho^{\text{in}}\right)\varphi - \int_0^t \int_{\mathbb{R}^d} \rho(s)\left(I(\varphi) - \mathcal{K}_c * (\rho(s)\cdot\nabla\varphi)\right)\mathrm{d}s$$
$$- \int_0^t \iint_{\mathbb{R}^{2d}} \mathcal{K}_0(x-y)(\nabla\varphi(x) - \nabla\varphi(y))\rho(s,\mathrm{d}x)\rho(s,\mathrm{d}y)\,\mathrm{d}s.$$

Since ρ_ε solves (6.45), it holds for any $t > 0$

$$\mathcal{F}_\varepsilon(\rho_\varepsilon, t) = 0,$$

where $\mathcal{F}_\varepsilon$ is the same functional as $\mathcal{F}$ with ∇K replaced with κ_ε. So that for any $t \in [0,T]$

$$|\mathcal{F}(\rho,t)| \le |\mathcal{F}(\rho,t) - \mathcal{F}_\eta(\rho,t)| + |\mathcal{F}_\eta(\rho,t) - \mathcal{F}_\eta(\rho_\varepsilon,t)| + |\mathcal{F}_\eta(\rho_\varepsilon,t) - \mathcal{F}_\varepsilon(\rho_\varepsilon,t)|.$$

But note that for $\eta > \varepsilon \ge 0$

$$|\kappa_\varepsilon(x) - \kappa_\eta(x)| \le \mathbb{1}_{\varepsilon\le|x|\le\eta}|x|^{1-\mathrm{a}} \le \eta|x|^{-\mathrm{a}}.$$

We deduce that for any $\varrho \in L^1([0,T];L^{p_\mathrm{a}})$, by (6.3), it holds

$$|\mathcal{F}_\eta(\varrho,t) - \mathcal{F}_\varepsilon(\varrho,t)| \le \eta \int_0^t \iint |x-y|^{-\mathrm{a}}\varrho_s(\mathrm{d}x)\varrho_s(\mathrm{d}y)\,\mathrm{d}s$$
$$\le \eta\, \mathcal{C}^{\text{HLS}}_{d,\mathrm{a},p_{\mathrm{a}/2}} \int_0^t \|\varrho\|^2_{L^{\frac{2d}{2d-\mathrm{a}}}}\,\mathrm{d}s \le \eta\, \mathcal{C}^{\text{HLS}}_{d,\mathrm{a},p_{\mathrm{a}/2}} \int_0^t \|\varrho\|_{L^{p_\mathrm{a}}}\,\mathrm{d}s.$$

So that

$$|\mathcal{F}(\rho,t)| \le \eta\, \mathcal{C}^{\text{HLS}}_{d,\mathrm{a},p_{\mathrm{a}/2}} \left(\int_0^t \|\rho\|_{L^{p_\mathrm{a}}}\,\mathrm{d}s + \sup_{0<\varepsilon<1} \int_0^t \|\rho_\varepsilon\|_{L^{p_\mathrm{a}}}\,\mathrm{d}s\right)$$
$$+ |\,\mathcal{F}_\eta(\rho,t) - \mathcal{F}_\eta(\rho_\varepsilon,t)|.$$

Letting first ε go to 0 makes the second term in the r.h.s. vanish, since for fixed $\eta > 0$, $\mathcal{F}_\eta$ is a smooth function on $L^1([0,T];L^{p_\mathrm{a}})$ and ρ_ε goes weakly to ρ as ε goes to 0, then letting η go to 0 yields $\mathcal{F}(\rho,t) = 0$, and ρ is a solution to the (FKS) equation in the sense of Definition 6.1.

Step 3. Uniqueness of the limiting point. We now show that there exists at most one such solution. Let ρ, $\tilde{\rho} \in L^1([0,T], L^p) \cap L^\infty([0,T], L^1_k)$ for some $p \geq p_{\rm a}$ and $T > 0$ be two solutions to the (FKS) equation with initial condition $\rho^{\rm in}$. We argue by a coupling argument. Define

$$X_t := X_0 - \lambda \int_0^t \int_{\mathbb{R}^d} \nabla K(X_s - y)\rho(\mathrm{d}y)\,\mathrm{d}s + Z_t^\alpha$$
$$Y_t := X_0 - \lambda \int_0^t \int_{\mathbb{R}^d} \nabla K(Y_s - y)\tilde{\rho}(\mathrm{d}y)\,\mathrm{d}s + Z_t^\alpha.$$

Due to the L^p regularity of ρ and $\tilde{\rho}$ and Lemma 6.6, $\nabla K * \rho$ and $\nabla K * \tilde{\rho}$ are Lipschitz if $p > p_{\rm a}$ and log-Lipschitz if $p = p_{\rm a}$. But $\mu(t) := \mathcal{L}(X_t)$ solves the linear PDE

$$\partial_t \mu = I(\mu) + \lambda \operatorname{div}\left((\nabla K * \rho)\mu\right),$$

for the initial condition $\mu(0) = M_0^{-1}\rho^{\rm in}$. By uniqueness of solution to this linear PDE with Lipschitz or log-Lipschitz coefficient, $\mathcal{L}(X_t) = M_0^{-1}\rho(t)$ (respectively $\mathcal{L}(Y_t) = M_0^{-1}\tilde{\rho}(t)$). Denoting $Z_s = X_s - Y_s$, and $\pi_s = \mathcal{L}(X_s, Y_s)$ yields

$$|Z_t|^2 = -2\lambda \int_0^t \iint_{\mathbb{R}^{2d}} Z_s \cdot (\nabla K(X_s - x) - \nabla K(Y_s - y))\pi_s(\mathrm{d}x, \mathrm{d}y)\,\mathrm{d}s.$$

Introducing $\bar{X}_s$ i.i.d. from X_s (respectively $\bar{Y}_s$ i.i.d. from Y_s) and taking the expectation yields

$$\begin{aligned}\mathbb{E}\left[|Z_t|^2\right] &\leq 2\lambda \int_0^t \mathbb{E}\left[|Z_s||\nabla K(X_s - \bar{X}_s) - \nabla K(Y_s - \bar{Y}_s)|\right]\mathrm{d}s \\ &\leq \begin{cases} C\int_0^t (\|\rho\|_{L^p} + \|\tilde{\rho}\|_{L^p} + 2)\,\mathbb{E}\left[|Z_s|^2\right]\mathrm{d}s, & \text{if } p > p_{\rm a} \\ C\int_0^t (\|\rho\|_{L^{p_{\rm a}}} + \|\tilde{\rho}\|_{L^{p_{\rm a}}} + 2)\,\mathbb{E}\left[|Z_s|^2\right]\left(1 + \frac{\ln^-\left(\mathbb{E}\left[|Z_s|^2\right]\right)}{2}\right)\mathrm{d}s & \text{else.}\end{cases}\end{aligned}$$

where we used Lemma 6.6. By Gronwall's inequality, we get

$$\forall t \in [0,T], \mathbb{E}\left[|Z_t|^2\right] = 0, \text{ i.e. } \forall t \in [0,T], \rho(t) = \tilde{\rho}(t),$$

which yields the desired results. □

6.4 Proof of Theorem 6.7

We first study the local and asymptotic space behavior of the fractional Laplacian of some basic functions.

Lemma 6.7. *Let $\varphi \in C_c^\infty$ be such that $\int_{\mathbb{R}^d} \varphi = 1$. Then for any $\mathrm{a} > \alpha$*

$$|I(|x|^{\mathrm{a}}\varphi)| \leq C\,\langle x \rangle^{-(d+\alpha)}. \tag{6.46}$$

Proof. Let $\varphi_{\mathrm{a}} := |x|^{\mathrm{a}}\varphi$ and $R > 0$ be such that $\mathrm{supp}(\varphi) \subset B_R$. Then, for any $x \in B_R^c$, we obtain

$$I(\varphi_{\mathrm{a}})(x) = \int_{B_R} \frac{\varphi_{\mathrm{a}}(y)\,\mathrm{d}y}{|x-y|^{d+\alpha}} \in \left(\frac{m_{\varphi_{\mathrm{a}}}}{(|x|+R)^{d+\alpha}}, \frac{m_{\varphi_{\mathrm{a}}}}{(|x|-R)^{d+\alpha}} \right). \tag{6.47}$$

Now, assume $x \in B_r$ for a given $r > R$. Then we write the fractional Laplacian as

$$I(\varphi_{\mathrm{a}}) = \int_{\mathbb{R}^d} h_{\alpha,\mathrm{a}}(y)\,\mathrm{d}y,$$

where

$$h_{\alpha,\mathrm{a}}(y) = \frac{\varphi_{\mathrm{a}}(y) - \varphi_{\mathrm{a}}(x)}{|x-y|^{d+\alpha}} \qquad \text{when } \alpha \in (0,1)$$
$$h_{\alpha,\mathrm{a}}(y) = \frac{\varphi_{\mathrm{a}}(y) - \varphi_{\mathrm{a}}(x) - (y-x)\cdot\nabla\varphi_{\mathrm{a}}(x)}{|x-y|^{d+\alpha}} \qquad \text{when } \alpha \in [1,2).$$

Then since $\varphi_{\mathrm{a}} \in W^{\mathrm{a},\infty}$, we obtain that $h_{\alpha,\mathrm{a}}(y) \le C|x-y|^{-d+\mathrm{a}-\alpha}$, which, since $\mathrm{a} > \alpha$, implies that $h_{\alpha,\mathrm{a}} \in L^1_{\mathrm{loc}}$. Moreover, when $|y| > r$, then

$$h_{\alpha,\mathrm{a}}(y) \le \frac{C_\varphi r^\alpha}{(|y|-r)^{d+\alpha}} \in L^1(B_r^c).$$

Therefore, $h_{\alpha,\mathrm{a}} \in L^1$ uniformly in $x \in B_r$. Hence $I(\varphi_{\mathrm{a}}) \in L^\infty(B_r)$, which, combined with (6.47), leads to the expected result. □

Lemma 6.8. *Let $\varphi \in C_c^\infty$ be such that $\int_{\mathbb{R}^d} \varphi = 1$ and $\mathbb{1}_{B_r} \le \varphi \le \mathbb{1}_{B_{2r}}$. Then for any $k \in (0,\alpha)$*

$$\left| I(|x|^k \varphi^c) \right| \le C \langle x \rangle^{k-\alpha}, \tag{6.48}$$

where $\varphi^c = 1 - \varphi$.

Proof. The proof is a straightforward adaptation of [31, Remark 4.2] for $k > 1$ and [132, Proposition 2.2] for $k < 1$. □

We are now ready to prove the finite time blow-up.

Proof. (**Proof of Theorem 6.7.**) Let $\varphi \in C_c^\infty(\mathbb{R})$ even and nonincreasing be such that $\int_{\mathbb{R}} \varphi = 1$ and $\mathbb{1}_{B_r} \le \varphi \le \mathbb{1}_{B_{2r}}$ for a given $r \in (0, 1/2)$ and $\varphi^c = 1 - \varphi$. We define

$$m(x) := \varphi(|x|)|x|^{\mathrm{a}} + \varphi^c(|x|)|x|^k.$$

Assuming the existence of $\rho \in L^\infty((0,T), L^1_k)$ to the (FKS) equation, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d} \rho m \right) &= \int_{\mathbb{R}^d} \rho\, I(m) - \lambda \iint_{\mathbb{R}^{2d}} \frac{(\nabla m(x) - \nabla m(y))\cdot(x-y)}{|x-y|^{\mathrm{a}}} \rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &= \mathcal{I}_1 - \lambda \mathcal{I}_2, \end{aligned} \tag{6.49}$$

- ***Estimate of $\mathcal{I}_1$.*** By the inequalities (6.46) and (6.48), we get

$$I(m) \le C \langle x \rangle^{k-\alpha}. \tag{6.50}$$

Hence, for some constant $C_1 > 0$, the following inequality holds

$$\mathcal{I}_1 \leq C \int_{\mathbb{R}^d} \langle x \rangle^{k-\alpha} \rho \leq C_1 M_0.$$

• ***Estimate of*** $\mathcal{I}_2$***.***

◇ *Step one: case* $1 < k < \alpha < \mathrm{a}$. In this case, by convexity we have for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$

$$(\nabla m(x) - \nabla m(y)) \cdot (x - y) = g(x, y) - h(x, y)\, x \cdot y \geq 0,$$

with $m'(|x|) = \nabla m(x) \cdot \frac{x}{|x|}$ and

$$\begin{aligned} g(x, y) &= m'(|x|)|x| + m'(|y|)|y| \\ h(x, y) &= m'(|x|)|x|^{-1} + m'(|y|)|y|^{-1}. \end{aligned}$$

Since $|x - y|^{\mathrm{a}} \leq 2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})$, we obtain

$$\begin{aligned} \mathcal{I}_2 = \iint_{\mathbb{R}^{2d}} \frac{g(x, y) - h(x, y)\,(x \cdot y)}{|x - y|^{\mathrm{a}}} \rho(\mathrm{d}x)\rho(\mathrm{d}y) &\geq \iint_{\mathbb{R}^{2d}} \frac{g(x, y)}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})} \rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\quad - \iint_{\mathbb{R}^{2d}} \frac{h(x, y) x \cdot y}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})} \rho(\mathrm{d}x)\rho(\mathrm{d}y). \end{aligned}$$

Next since

$$\iint_{\mathbb{R}^{2d}} \frac{h(x, y) x \cdot y}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})} \rho(\mathrm{d}x)\rho(\mathrm{d}y) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{h(x, y) x \cdot y}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})} \rho(\mathrm{d}x) \right) \rho(\mathrm{d}y),$$

by Fubini's theorem, and since for any $y \in \mathbb{R}^d$ the map $x \mapsto \frac{h(x,y) x \cdot y}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})}$ is odd and ρ is even, we get

$$\mathcal{I}_2 \geq \iint_{\mathbb{R}^{2d}} \frac{g(x, y)}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})} \rho(\mathrm{d}x)\rho(\mathrm{d}y). \tag{6.51}$$

We remark that if $(x, y) \in B_r^2$,

$$\frac{g(x, y)}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})} = \frac{\mathrm{a}}{2^{\mathrm{a}}}.$$

If $(x, y) \in (B_{2r}^c)^2$,

$$\frac{g(x, y)}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})} = \frac{k(|x|^k + |y|^k)}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})} \geq \frac{k(2r)^{\mathrm{a}-k}}{2^{\mathrm{a}}(|x||y|)^{\mathrm{a}-k}}.$$

If $(x, y) \in B_r \times B_{2r}^c$,

$$\frac{g(x, y)}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})} = \frac{\mathrm{a}|x|^{\mathrm{a}} + k|y|^k}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})} \geq \frac{k|y|^k}{2^{\mathrm{a}}(r + |y|^{\mathrm{a}})}.$$

Moreover, when $x \in B_{2r} \backslash B_r$,

$$m'(|x|)|x| = \varphi'(|x|)(|x|^{\mathrm{a}+1} - |x|^{k+1}) + \mathrm{a}\varphi(|x|)|x|^{\mathrm{a}} + k\varphi^c(|x|)|x|^k.$$

Remarking that we can take φ decreasing and $r < 1/2$, which implies that $|x| \le 1$ and

$$m'(|x|)|x| \ge \mathrm{a}\varphi(|x|)|x|^{\mathrm{a}} + k\varphi^c(|x|)|x|^k \ge k|x|^{\mathrm{a}},$$

it allows us to do the same kind of estimates for the remaining $(x, y) \in \mathbb{R}^{2d}$ and obtain

$$\frac{g(x,y)}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})} \ge C \langle x \rangle^{k-\mathrm{a}} \langle y \rangle^{k-\mathrm{a}} . \tag{6.52}$$

Combining (6.50), (6.51) and (6.52), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^d} \rho\, m\right) &\le C_1 \int_{\mathbb{R}^d} \rho \langle x \rangle^{k-\alpha} - C_2 \lambda \iint_{\mathbb{R}^{2d}} \langle x \rangle^{k-\mathrm{a}} \langle y \rangle^{k-\mathrm{a}} \rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\le C_1 M_{k-\alpha} - C_2 \lambda M_{k-\mathrm{a}}^2, \end{aligned} \tag{6.53}$$

where $M_k = \int_{\mathbb{R}^d} \rho \langle x \rangle^k$. We define

$$Y := M_0 + \int_{\mathbb{R}^d} \rho m = \int_{\mathbb{R}^d} \rho\,(1+m)\,.$$

Remarking that

$$\frac{1}{2}(1+m) \le \langle x \rangle^k \le 2^{k/2}(1+m),$$

we obtain that Y can always be compared to M_k up to a constant depending on k. Therefore, Hölder's inequality yield

$$M_0 \le M_{k-\mathrm{a}}^{\frac{k}{\mathrm{a}}} M_k^{1-\frac{k}{\mathrm{a}}} \le C M_{k-\mathrm{a}}^{\frac{k}{\mathrm{a}}} Y^{1-\frac{k}{\mathrm{a}}}.$$

Thus, using the fact that $M_{k-\alpha} < M_0$ because $k - \alpha < 0$ and the conservation of the total mass M_0, we obtain

$$\frac{\mathrm{d}Y}{\mathrm{d}t} \le C_1\, M_0 - C_2'\, \lambda\, M_0^{\frac{2\mathrm{a}}{k}} Y^{2\left(1-\frac{\mathrm{a}}{k}\right)}.$$

By assumption (6.10) for the appropriate C^*,

$$\varepsilon := C_2 \left(1 - \frac{C_1 Y^{2(\frac{\mathrm{a}}{k}-1)}(0)}{C_2 \lambda M_0^{\frac{2\mathrm{a}}{k}-1}}\right) > 0.$$

Then for any $t \ge 0$, $\frac{\mathrm{d}Y}{\mathrm{d}t} \le 0$ and

$$Y^{2(\frac{\mathrm{a}}{k}-1)}(t) \le Y^{2(\frac{\mathrm{a}}{k}-1)}(0) = \frac{C_2 - \varepsilon}{C_1} \lambda M_0^{\frac{2\mathrm{a}}{k}-1},$$

and

$$\frac{\mathrm{d}Y}{\mathrm{d}t} \le -\varepsilon \lambda M_0^{\frac{2\mathrm{a}}{k}} Y^{2\left(1-\frac{\mathrm{a}}{k}\right)}.$$

By Gronwall's inequality, we deduce

$$Y(t) \leq \left(Y(0)^{\frac{2\mathrm{a}}{k}-1} - \varepsilon\lambda \left(\tfrac{2\mathrm{a}}{k} - 1 \right) M_0^{\frac{2\mathrm{a}}{k}} t \right)^{\frac{k}{2\mathrm{a}-k}}.$$

Since Y is positive and the above inequality goes to 0 in finite time, we deduce that the solution ceases to be well defined in L^1 in a finite time T^* verifying

$$T^* < \frac{kY(0)^{\frac{2\mathrm{a}}{k}-1}}{\varepsilon\lambda(2\mathrm{a}-k)M_0^{\frac{2\mathrm{a}}{k}}} = \frac{k}{2\mathrm{a}-k} \frac{Y(0)^{\frac{2\mathrm{a}}{k}-1}}{C_2\lambda M_0^{\frac{2\mathrm{a}}{k}} - C_1 Y^{2(\frac{\mathrm{a}}{k}-1)}(0) M_0},$$

which proves the result.

◇ *Step two: Case* $0 < k < \alpha \leq 1 \leq \mathrm{a}$. We use the symmetry between x and y to rewrite

$$\begin{aligned}\mathcal{I}_2 \geq C &\left(\iint_{\substack{|x|\leq r\\ |y|\leq r}} + \iint_{\substack{|x|>r\\ |x-y|\leq r/2}} + \iint_{\substack{|x|>r\\ |x-y|>r/2\\ |y|<r}} + \iint_{\substack{|x|>r\\ |x-y|>r/2\\ |y|>r}} \right) \\ &\qquad \frac{(\nabla m(x) - \nabla m(y)) \cdot (x-y)}{|x-y|^{\mathrm{a}}} \rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &= \mathcal{I}_2^1 + \mathcal{I}_2^2 + \mathcal{I}_2^3 + \mathcal{I}_2^4.\end{aligned}$$

• Estimate of $\mathcal{I}_2^1$. For $|x| \leq r$, we have $m(x) = |x|^{\mathrm{a}}$. Hence by strict convexity (since $\mathrm{a} \geq 1$), we expand the inner product similarly as in the beginning of step one to obtain, with the same arguments

$$\begin{aligned}\mathcal{I}_2^1 &= \iint_{\substack{|x|\leq r\\ |y|\leq r}} \frac{g(x,y) - h(x,y)\,(x \cdot y)}{|x-y|^{\mathrm{a}}} \rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\geq \iint_{\substack{|x|\leq r\\ |y|\leq r}} \frac{g(x,y) - h(x,y) x \cdot y}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})} \rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &= \iint_{\substack{|x|\leq r\\ |y|\leq r}} \frac{g(x,y)}{2^{\mathrm{a}}(|x|^{\mathrm{a}} + |y|^{\mathrm{a}})} \rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\geq \frac{\mathrm{a}}{2^{\mathrm{a}}} \left(\int_{B_r} \rho \right)^2.\end{aligned}$$

• Estimate of $\mathcal{I}_2^2$. We may choose the linking function φ in the definition of m smooth enough so that for $|x| > r$ it holds $|\nabla m(x)| \leq C|x|^{k-1}$. And since $k-1 \leq 0$ and $1-\mathrm{a} \leq 0$, we have

$$\begin{aligned}\mathcal{I}_2^2 &= \iint_{\substack{|x|>r\\ |y|>r\\ |x-y|>r/2}} \frac{(\nabla m(x) - \nabla m(y)) \cdot (x-y)}{|x-y|^{\mathrm{a}}} \rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\geq -C \iint_{\substack{|x|>r\\ |y|>r\\ |x-y|>r/2}} \left(|x|^{k-1} + |y|^{k-1} \right) |x-y|^{1-\mathrm{a}} \rho(\mathrm{d}x)\rho(\mathrm{d}y) \\ &\geq -2^{\mathrm{a}} C r^{k-\mathrm{a}} \left(\int_{B_r^c} \rho \right)^2,\end{aligned}$$

- Estimate of $\mathcal{I}_2^3$. Similar considerations yield

$$\mathcal{I}_2^3 = \iint_{\substack{|x|>r \\ |y|\leq r \\ |x-y|>r/2}} \frac{(\nabla m(x) - \nabla m(y)) \cdot (x-y)}{|x-y|^{\mathrm{a}}} \rho(\mathrm{d}x)\rho(\mathrm{d}y)$$
$$\geq -2^{\mathrm{a}-1} C \left(r^{k-\mathrm{a}} + 1\right) \left(\int_{B_r^c} \rho\right) \left(\int_{B_r} \rho\right).$$

- Estimate of $\mathcal{I}_2^4$. When $|x-y| \leq r/2$ and $|x| > r$, remark that it holds

$$|x| \;\leq\; |x-y| + |y| \;\leq\; \frac{r}{2} + |y| \;<\; \frac{|x|}{2} + |y|$$
$$|y| \;\leq\; |x-y| + |x| \;\leq\; \frac{r}{2} + |x| \;\leq\; \frac{3|x|}{2}$$

which implies that $r \leq |x| \leq 2\,|y| \leq 3\,|x|$. Therefore, we can write

$$\mathcal{I}_2^4 = \iint_{\substack{|x|>r \\ |x-y|\leq r/2}} \frac{(\nabla m(x) - \nabla m(y)) \cdot (x-y)}{|x-y|^{\mathrm{a}}} \rho(\mathrm{d}x)\rho(\mathrm{d}y)$$
$$\geq -C \iint_{\substack{|x|>r \\ |x-y|\leq r/2 \\ |y|>r/2}} \left| |x|^{k-2} x - |y|^{k-2} y \right| |x-y|^{1-\mathrm{a}} \, \rho(\mathrm{d}x)\rho(\mathrm{d}y).$$

Then, since $\left|\nabla |z|^{k-2}\right| = |k-2|\,|z|^{k-3}$ and $2\,|x-y| \leq |x|$, we obtain

$$\left| |x|^{k-2} x - |y|^{k-2} y \right| \leq \left| \left(|x|^{k-2} - |y|^{k-2} \right) x \right| + \left| |y|^{k-2} (x-y) \right|$$
$$\leq C_k\, |x|\, |x-y| \sup_{|z|\geq |x|/2} |z|^{k-3} + |y|^{k-2} |x-y|$$
$$\leq C_{\mathrm{a},k}\, r^{k-\mathrm{a}}\, |x-y|^{\mathrm{a}-1},$$

from which we get

$$\mathcal{I}_2^2 \geq -C_{\mathrm{a},k} r^{k-\mathrm{a}} \left(\int_{B_{r/2}^c} \rho \right)^2.$$

Defining $Y = \int_{\mathbb{R}^d} \rho m$ and using the fact that

$$\tilde{Y} := \int_{B_r^c} \rho \;\leq\; C_{m,r}\, Y \;\text{ and }\; \int_{B_r} \rho \;=\; M_0 - \tilde{Y},$$

and gathering the previous estimates yields the existence of positive constants C_2, C_3, depending on a, k and r such that

$$\mathcal{I}_2 \geq \frac{\mathrm{a}}{2^{\mathrm{a}}} \left(M_0 - \tilde{Y} \right)^2 - C_{\mathrm{a},k} r^{k-\mathrm{a}} \tilde{Y}^2 - C 2^{\mathrm{a}} r^{k-\mathrm{a}} \tilde{Y}^2 - C 2^{\mathrm{a}-1} \left(r^{k-\mathrm{a}} + 1 \right) \tilde{Y} \left(M_0 - \tilde{Y} \right)$$
$$\geq \frac{C_2}{2} \left(M_0^2 - C_3 \tilde{Y}^2 \right).$$

Coming back to (6.49) and using the fact that $\tilde{Y} \leq C_{m,r}Y$ yields the existence of a constant C_4 such that

$$\frac{\mathrm{d}Y}{\mathrm{d}t} \leq C_1 M_0 + \frac{C_2}{2}\lambda \left(C_4 Y^2 - M_0^2 \right).$$

In particular, as long as $Y^2 \leq (2\,C_4)^{-1} M_0^2$ and $C_2 \lambda M_0 \geq 8\,C_1$ it holds

$$\frac{\mathrm{d}Y}{\mathrm{d}t} \leq C_1 M_0 - \frac{C_2}{4}\lambda M_0^2 \leq -C_1 M_0 \leq 0. \tag{6.54}$$

In particular, if $Y(0)^2 \leq (2\,C_4)^{-1} M_0^2$ then Y remains decreasing for all times and for all $t > 0$, $Y(0)^2 \leq (2C_3)^{-1} M_0^2$. By using again (6.54), this implies

$$Y(t) \leq Y(0) - C_1 M_0\, t,$$

which becomes negative in finite time and leads again to a contradiction. The fact that the condition (6.11) is sufficient comes from the fact that there exists a constant $C > 0$ such that

$$Y = \int_{\mathbb{R}^d} \rho m \leq C \int_{\mathbb{R}^d} \rho(x) |x|^k \,\mathrm{d}x,$$

since $k <$ a. □

Acknowledgements

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Chapter 7

p-Laplacian Keller-Segel Equation: Fair Competition and Diffusion Dominated Cases

Abstract

This work deals with the aggregation diffusion equation

$$\partial_t \rho = \Delta_p \rho + \lambda \operatorname{div}((\nabla K * \rho)\rho),$$

where $\nabla K(x) = \frac{x}{|x|^{\mathrm{a}}}$ is an attraction kernel and Δ_p is the so called p-Laplacian. We show that the domain $\mathrm{a} < p(d+1) - 2d$ is subcritical with respect to the competition between the aggregation and diffusion by proving that there is existence unconditionally with respect to the mass. In the critical case we show existence of solution in a small mass regime for an $L \ln L$ initial condition.

Résumé

Ce travail concerne l'étude d'une famille d'équations d'agrégation diffusion

$$\partial_t \rho = \Delta_p \rho + \lambda \operatorname{div}((\nabla K * \rho)\rho),$$

où $\nabla K(x) = \frac{x}{|x|^{\mathrm{a}}}$ est un champ d'attraction et Δ_p est le p-Laplacien. On montre que le domaine $\mathrm{a} < p(d+1) - 2d$ est sous-critique du point de vue de la compétition entre l'agrégation et la diffusion en montrant l'existence de solution quelle que soit la masse. Dans le cas critique, on montre l'existence de solution dans un régime de petite masse pour une condition $L \ln L$.

Contents

7.1 Version francaise abrégée

On entend par équation d'agrégation-diffusion une equation aux dérivées partielles non linéaire sur $\mathbb{R}^d$ de la forme

$$\partial_t \rho = \mathsf{D}(\rho) + \lambda \operatorname{div}\left((\nabla K * \rho)\rho\right),$$

où pour $\mathrm{a} \in (0,d)$, $\nabla K(x) = \frac{x}{|x|^{\mathrm{a}}}$ est un noyau d'attraction, $\lambda > 0$ indique l'intensité de cette interaction et D est un opérateur de diffusion. Cette équation décrit par exemple l'évolution de la densité d'une population de bactéries ou d'astres en gravitation (voir par exemple [120]).

Ce modèle a été largement étudié dans le cas de l'opérateur de diffusion non-linéaire $\mathsf{D}(\rho) = \Delta(\rho^m)$ pour $m > 0$ (voir [54]). Grâce à la structure algébrique conférée par ce choix de diffusion, on montre que l'EDP est en fait un flot de gradient pour la distance de Wasserstein d'ordre 2 d'une certaine fonctionnelle. De l'étude de cette fonctionnelle découle que la ligne $\mathrm{a} = 2 - d(m-1)$ (dans le plan (m, a)) est critique du point de vue de la compétition entre l'agrégation et la diffusion. Le demi-plan situé au dessus de cette droite correspond au régime d'agrégation dominante, et celui au dessous à celui de diffusion dominante

Lorsque la diffusion est fractionnaire i.e. $\mathsf{D} = \Delta^{\alpha/2}$ est le Laplacien fractionnaire d'exposant $\alpha \in (0,2)$, on montre que la ligne critique est la première bissectrice $\mathrm{a} = \alpha$ (voir [189, 132]), et qu'elle délimite dans ce cas également deux régimes opposés.

Ce chapitre poursuit cette étude, dans le cas du p-Laplacien

$$\mathsf{D}(\rho) = \Delta_p(\rho) = \operatorname{div}(|\nabla\rho|^{p-2}\nabla\rho)$$

pour $p \in \left(\frac{2d}{d+1}, \frac{3d}{d+1}\right)$. On montre ici que le domaine $\mathrm{a} < p(d+1) - 2d$ est sous-critique et qu'il y a existence pour petite masse dans le cas d'égalité. Au passage on établit une estimation de moments pour la p-équation de la chaleur.

7.2 Introduction

Aggregation diffusion equations play an important role in the modeling of collective behavior and more specially, in the case of the motion of cells and bacteria (see for instance [120]). The (parabolic-elliptic) Keller-Segel equation, which has been extensively studied (see [40]), is a typical example. In generality, we mean by aggregation equation the class of mean field nonlinear conservation equation of the form

$$\partial_t \rho = \mathsf{D}(\rho) + \lambda \operatorname{div}\left((\nabla K * \rho)\rho\right), \tag{7.1}$$

where ∇K is an aggregation kernel defined as $\nabla K(x) = \frac{x}{|x|^{\mathrm{a}}}$, $\lambda > 0$ is a parameter encoding the intensity of the aggregation and D is some diffusion operator. Equation (7.1) can then be interpreted as the evolution of the probability density of particles attracting each other through ∇K and diffusing through D. Then depending on the result of the competition between these two phenomena, the equation may yield to global existence or finite time blow up.

The case of power law diffusion $\mathsf{D}(\rho) = \Delta(\rho^m)$ for some $m > 0$, has been studied in [54] where the line $\mathrm{a} = 2 - d(m-1)$ is shown to be critical. In that case equation (7.1) can be seen as the gradient flow of some suitable functional with respect to the Wasserstein-2 distance and the criticality appears from the asymptotic study of this functional.

The case of fractional diffusion $\mathsf{D}(\rho) = \Delta^{\alpha/2}\rho$ for some $\alpha \in (0,2)$ has been studied in [189, 132], where it is shown that the critical line is the first bisector $\alpha = \mathrm{a}$, above which blow up of solutions may occur in finite time, and under which global well-posedness and propagation of chaos hold.

In order to complete this study, this chapter investigates the case where the diffusion operator is the p-Laplacian $\mathsf{D} = \Delta_p$ (see e.g. [150]), which is defined for any $\rho \in W^{1,p-1}_{\mathrm{loc}}$ by

$$\forall \varphi \in C_c^\infty(\mathbb{R}^d),\ \langle \Delta_p \rho, \varphi \rangle = -\int_{\mathbb{R}^d} |\nabla \rho|^{p-2}\, \nabla \rho \cdot \nabla \varphi,$$

and appears for example in the diffusion equations for sandpiles (see e.g. [10, 91]).

7.3 Main results

The aggregation equation (7.1) with $\mathsf{D} = \Delta_p$,

$$\partial_t \rho = \operatorname{div}(|\nabla \rho|^{p-2} \nabla \rho) + \lambda \operatorname{div}((\nabla K * \rho)\rho), \tag{7.2}$$

has not been much studied, to the best of the author's knowledge. The only reference at this matter is [154], which concerns the case $\mathrm{a} = d$ and $p \in (2, \frac{3d}{d+1})$.

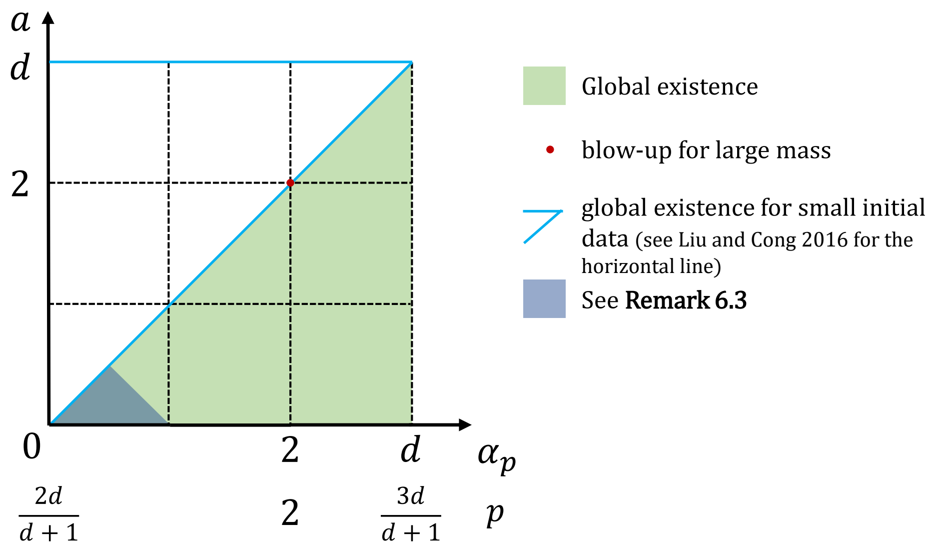

Figure 7.1: Graph of results about equation (7.2). The case $\mathrm{a} + \alpha_p < 1$ has to be a priori excluded, since moments of order at least $1-\mathrm{a}$ on ρ are needed to make sense of the term $(\nabla K * \rho)$, and Δ_p propagates at most moments of order α_p (see Lemma 7.3). We refer to Remark 6.3.
Graphique des résultats pour l'équation (7.2). Le cas $\mathrm{a} + \alpha_p < 1$ doit être priori exclu, puisque des moments d'ordre au moins $1-\mathrm{a}$ sur ρ sont requis pour donner du sens au terme $(\nabla K * \rho)$, et Δ_p propage des moments d'ordre au plus α_p (voir Lemme 7.3). On renvoie à la Remarque 6.3.

Denoting $\|\rho\|_{L^p_k} := \|\rho m\|_{L^p}$ with $m(x) = \langle x\rangle^k$ and $L\ln L = \{\rho \geq 0, \rho \in L^1, \rho\ln\rho \in L^1\}$, we state the main result of this chapter.

Theorem 7.1. *Let $d \geq 2$, $\lambda > 0$ and $(\mathrm{a}, p) \in (0, d)\times\left(\frac{2d}{d+1}, \frac{3d}{d+1}\right)$. Denote $\alpha_p := p(d+1)-2d$ and assume $\alpha_p + \mathrm{a} > 1$. Let $\rho^{\mathrm{in}} \in L\ln L \cap L^1_k$ for some $k \in ((1-\mathrm{a})_+, \alpha_p \wedge 1)$. Then in the*

- ***Diffusion dominated** case $\mathrm{a} < \alpha_p$*
- ***Fair competition** case $\mathrm{a} = \alpha_p$ if ρ^{in} satisfies*

$$M_0 := \|\rho^{\mathrm{in}}\|_{L^1} < C_{d,p}\lambda^{-\frac{1}{3-p}},$$

there exists a solution $\rho \in L^{p/p'}_{\mathrm{loc}}(\mathbb{R}_+, L^{p^*/p'}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^1_k)$ to equation (7.2) with initial condition ρ^{in}.

Remark 7.2. *The constant $C_{d,p}$ is given by*

$$C_{d,p} = \left((d-\alpha_p)\mathcal{C}^{\mathrm{HLS}}_{d,\alpha_p,\frac{2d}{2d-\alpha_p}}\left(p'^{-1}\mathcal{C}^{\mathrm{S}}_{d,p}\right)^p\right)^{-\frac{1}{3-p}},$$

where for $\mathrm{a} \in (0, d)$ and $2-\frac{\mathrm{a}}{d} = \frac{2}{q}$, $\mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},q}$ is the best constant for Hardy-Litllewood-Sobolev's inequality,

$$\iint_{\mathbb{R}^{2d}} |x-y|^{-\mathrm{a}}\rho(x)\rho(y)\,\mathrm{d}x\,\mathrm{d}y \leq C^{HLS}_{d,\mathrm{a},q}\|\rho\|^2_{L^q},$$

and for $q \in (0,d)$, and $q^* = dq/(d-q)$, $C^S_{d,q}$ is the best constant for Sobolev's embeddings,

$$\|\rho\|_{L^{q^*}} \le C^S_{d,q} \|\nabla \rho\|_{L^q}.$$

The explicit value for these constants are known (see [12, 203, 147]).

Note that for $d = 2$, the point $(\mathrm{a}, m) = (2,1)$ in the context of power law diffusion, $(\mathrm{a}, \alpha) = (2,2)$ in the notations of fractional diffusion and $(\mathrm{a}, p) = (2,2)$ in the notations of the present chapter all correspond to the classical Keller-Segel equation, and the three different definitions of the fair competition case coincide for this equation.

7.4 Proof of Theorem 7.1

We begin this section by introducing for $p > 1$, the p-Fisher information I_p on $(W^{1,p})^{p'} := \{\rho, \rho^{1/p'} \in W^{1,p}\}$ as

$$I_p(\rho) = \int_{\mathbb{R}^d} \frac{|\nabla \rho|^p}{\rho} = (p')^p \|\nabla(\rho^{1/p'})\|^p_{L^p},$$

which is a generalization of the classical Fisher information (i.e. the case $p = 2$). First remark that a straightforward computation using Hölder's and Sobolev's inequalities shows that $(W^{1,p})^{p'} \subsetneq W^{1,p-1}_{\mathrm{loc}}$ so that $\Delta_p \rho$ is well-defined for ρ with finite p-Fisher information. Then for any $p \in \left(\frac{2d}{d+1}, \frac{3d}{d+1}\right)$, $q \in [1, r]$ with $r = \frac{p^*}{p'}$ and $\rho \in (W^{1,p})^{p'} \cap L^1$ it holds

$$\|\rho\|_{L^q} \le ((p')^{-1} \mathcal{C}^{\mathrm{S}}_{d,p})^{\frac{r'p'}{q'}} \|\rho\|_{L^1}^{1-\frac{r'}{q'}} I_p(\rho)^{\frac{r'p'}{q'p}}. \tag{7.3}$$

Indeed by Sobolev's embeddings it holds

$$\|\rho\|^{p/p'}_{L^r} = \|\rho^{1/p'}\|^p_{L^{p^*}} \le (C^S_{d,p})^p \|\nabla(\rho^{1/p'})\|^p_{L^p} \le ((p')^{-1} C^S_{d,p})^p I_p(\rho),$$

and using interpolation between L^1 and L^r yields the result. Then we need some tools in order to provide some moments estimate.

Lemma 7.3. *There is $C > 0$ such that for any $\rho \in L^1 \cap (W^{1,p})^{p'}$ and $k > 0$ it holds*

$$\int_{\mathbb{R}^d} (\Delta_p \rho) m \le C \begin{cases} \|\rho\|_{L^1}^{\frac{p-1}{\alpha_p}} I_p(\rho)^{\frac{\alpha_p - 1}{\alpha_p}}, & \text{if } p \ge 2 \text{ and } k \in [0,1], \\ (\int_{\mathbb{R}^d} \rho m)^{\frac{1}{p'}} I_p(\rho)^{\frac{1}{p'}}, & \text{if } p \in \left(\frac{2d}{d+1}, 2\right) \text{ and } k \in (0, \alpha_p). \end{cases} \tag{7.4}$$

Proof. First in the case $p \ge 2$ and $k \in [0,1]$, since $k \le 1$, by Hölder's inequality it holds

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla \rho|^{p-2} \nabla \rho \cdot \nabla m &\le k \int_{\mathbb{R}^d} \rho^{\frac{1}{p'}} \rho^{\frac{-1}{p'}} |\nabla \rho|^{p-1} \langle x \rangle^{k-1} \\ &\le k \left(\int_{\mathbb{R}^d} \rho^{\frac{p}{p'}} \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} \frac{|\nabla \rho|^p}{\rho} \right)^{\frac{1}{p'}} = k \left(\|\rho\|_{L^{\frac{p}{p'}}} I_p(\rho) \right)^{\frac{1}{p'}}. \end{aligned}$$

Then, using inequality (7.3), we obtain

$$\|\rho\|_{L^{\frac{p}{p'}}} \leq C\|\rho\|_{L^1}^{\frac{1}{p'}\left(1-\frac{r'(p-2)}{(p-1)}\right)} I_p(\rho)^{\frac{r'p'}{(p-1)'p}},$$

and the result follows since

$$\left(\frac{r'p'}{(p-1)'\,p}+1\right)\frac{1}{p'} = \frac{\alpha_p-1}{\alpha_p} \text{ and } \frac{1}{p'}\left(1-\frac{r'(p-2)}{(p-1)}\right) = \frac{p-1}{\alpha_p}.$$

Then in the case $p \in \left(\frac{2d}{d+1}, 2\right)$ and $k \in (0, \alpha_p)$, by Hölder's inequality, since $p \leq 2$,

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla\rho|^{p-2}\,\nabla\rho\cdot\nabla m &\leq k\int_{\mathbb{R}^d} |\nabla\rho|^{p-1}\,\langle x\rangle^{k-1}\\ &\leq k\int_{\mathbb{R}^d} \rho^{\frac{1}{p'}}\,\langle x\rangle^{\frac{k}{p'}}\,\rho^{\frac{-1}{p'}}\,|\nabla\rho|^{p-1}\,\langle x\rangle^{\frac{k}{p}-1}\\ &\leq k\left(\int_{\mathbb{R}^d}\rho m\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^d}\frac{|\nabla\rho|^p}{\rho}\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^d}\langle x\rangle^{\frac{k-p}{2-p}}\right)^{\frac{2}{p}-1},\end{aligned}$$

and the result follows since by assumption $\frac{k-p}{2-p} < -d$. □

Proof of Theorem 7.1 We only provide the a priori estimate necessary to the rigorous proof. Following the claim of [154, Proof of Theorem 5.2, Step 1], we can retrieve well posedness for the regularized problem (7.1) with ∇K replaced with $\nabla K_\varepsilon(x) = \mathbb{1}_{|x|\geq\varepsilon}\nabla K(x) + \mathbb{1}_{|x|\leq\varepsilon}\varepsilon^{-\mathrm{a}}x$. The preservation of positivity is a consequence of Kato's inequality for the p-Laplacian (see [122, 155]). Then letting ε go to 0 and using the a priori estimate we are about to prove with together with a standard compactness argument (similarly as what is done in [40, Section 2.5]) provides the rigorous proof.

Step 1. Entropy dissipation. We first estimate the dissipation of entropy using together Hardy-Littlewood-Sobolev's inequality and (7.3) with $q = \frac{2d}{2d-\mathrm{a}}$ as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho\log\rho &= -\int_{\mathbb{R}^d}\left(|\nabla\rho|^{p-2}\,\nabla\rho\right)\cdot\nabla\log\rho + \lambda\int_{\mathbb{R}^d}\mathrm{div}((\nabla K*\rho)\rho)(\log\rho+1)\\ &= -I_p(\rho) + \lambda\int_{\mathbb{R}^d}(\mathrm{div}(\nabla K)*\rho)\rho\\ &= -I_p(\rho) + \lambda(d-\mathrm{a})\iint_{\mathbb{R}^{2d}}\frac{\rho(x)\rho(y)}{|x-y|^{\mathrm{a}}}\,\mathrm{d}x\,\mathrm{d}y\\ &\leq -I_p(\rho) + \lambda(d-\mathrm{a})\mathcal{C}^{\mathrm{HLS}}_{d,\mathrm{a},q}\left(p'^{-1}\mathcal{C}^{\mathrm{S}}_{d,p}\right)^{2\frac{r'p'}{q'}} M_0^{2-2\frac{r'}{q'}}\,I_p(\rho)^{2\frac{r'p'}{q'p}}.\end{aligned}$$

And since $2\frac{r'p'}{q'p} = \frac{\mathrm{a}}{\alpha_p}$, $2-2\frac{r'}{q'} = 2\left(1-(p-1)\frac{\mathrm{a}}{2\alpha_p}\right)$ and $2\frac{r'p'}{q'} = p\frac{\mathrm{a}}{\alpha_p}$, defining

$$C_{d,p}^{p-3} := (d-\alpha_p)\mathcal{C}^{\mathrm{HLS}}_{d,\alpha_p,\frac{2d}{2d-\alpha_p}}\left(p'^{-1}\mathcal{C}^{\mathrm{S}}_{d,p}\right)^p,$$

we conclude this step with

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho\log\rho \leq \begin{cases}-\left(1-\lambda C_{d,p}^{p-3}M_0^{3-p}\right)I_p(\rho), \text{ if } \mathrm{a}=\alpha_p\\ -\frac{1}{2}I_p(\rho)+C \text{ if } \mathrm{a}<\alpha_p.\end{cases} \tag{7.5}$$

Step 2. Moment estimate. First in the case $p \geq 2$, we choose $k \in ([1-\mathrm{a}]_+, 1)$ and use Lemma 7.3, symmetry and Young's inequality for any $\varepsilon > 0$ to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho m &\leq C M_0^{\frac{p-1}{\alpha_p}} I_p(\rho)^{\frac{\alpha_p-1}{\alpha_p}} - \frac{\lambda}{2}\iint_{\mathbb{R}^{2d}} \nabla K(x-y)\cdot(\nabla m(x)-\nabla m(y))\,\rho(\mathrm{d}x)\rho(\mathrm{d}y)\\ &\leq C_\varepsilon M_0^{\frac{p-1}{\alpha_p}\left(\frac{\alpha_p}{\alpha_p-1}\right)'} + \varepsilon I_p(\rho) - \frac{\lambda}{2}\iint_{\mathbb{R}^{2d}} \nabla K(x-y)\cdot(\nabla m(x)-\nabla m(y))\,\rho(\mathrm{d}x)\rho(\mathrm{d}y).\end{aligned}$$

Then in the case $p \in \left(\frac{2d}{d+1}, 2\right)$, we choose $k \in ([1-\mathrm{a}]_+, \alpha_p)$, use Lemma 7.3 and obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho m &\leq C\left(\int_{\mathbb{R}^d}\rho m\right)^{\frac{1}{p'}} I_p(\rho)^{\frac{1}{p'}} - \frac{\lambda}{2}\iint_{\mathbb{R}^{2d}} \nabla K(x-y)\cdot(\nabla m(x)-\nabla m(y))\,\rho(\mathrm{d}x)\rho(\mathrm{d}y)\\ &\leq C_\varepsilon\left(\int_{\mathbb{R}^d}\rho m\right)^{\frac{p}{p'}} + \varepsilon I_p(\rho) - \frac{\lambda}{2}\iint_{\mathbb{R}^{2d}} \nabla K(x-y)\cdot(\nabla m(x)-\nabla m(y))\,\rho(\mathrm{d}x)\rho(\mathrm{d}y).\end{aligned}$$

The last term in the r.h.s is dealt similarly as in [132, Proof of Proposition 3.1] and in any case, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho m \leq \varepsilon I_p(\rho) + C\left(1+\int_{\mathbb{R}^d}\rho m\right), \tag{7.6}$$

where C only depends on $d, p, \mathrm{a}, \lambda, \varepsilon, k$ and M_0.

Step 3. Conclusion. We will now only treat the case $\mathrm{a} = \alpha_p$ and $\lambda M_0^{3-p} C_{d,p}^{p-3} < 1$, since the case $\mathrm{a} < \alpha_p$ can be treated even more straightforwardly. For $k \geq 0$ denote $\nu_k > 0$ such that $\int_{\mathbb{R}^d} e^{-\nu_k m(x)}\,\mathrm{d}x = 1$, and recall that, with $h(u) = u\ln u - u + 1 \geq 0$, it holds

$$\int_{\mathbb{R}^d}\frac{\rho}{M_0}\ln\frac{\rho}{M_0} = \int_{\mathbb{R}^d} h\left(\frac{\rho}{M_0}e^{\nu_k m}\right)e^{-\nu_k m} + \int_{\mathbb{R}^d}\frac{\rho}{M_0}\ln(e^{-\nu_k m}) \geq -\nu_k\int_{\mathbb{R}^d}\frac{\rho}{M_0}m,$$

and then

$$\int_{\mathbb{R}^d}\rho\ln\rho \geq M_0\ln M_0 - \nu_k\int_{\mathbb{R}^d}\rho m,$$

which yields for fixed $\nu > \nu_k$, combining linearly (7.5) and (7.6)

$$\begin{aligned}(\nu-\nu_k)\int_{\mathbb{R}^d}\rho m &\leq -M_0\ln M_0 + \int_{\mathbb{R}^d}\rho\log\rho + \nu\int_{\mathbb{R}^d}\rho m\\ &\leq \int_{\mathbb{R}^d}\rho^{\mathrm{in}}\log\rho^{\mathrm{in}} + \nu\int_{\mathbb{R}^d}\rho^{\mathrm{in}} m + \nu\, C\int_0^t\left(\int_{\mathbb{R}^d}\rho(s)m + 1\right)\mathrm{d}s\\ &\quad - \left(1-\lambda C_{d,p}^{p-3}M_0^{3-p} - \varepsilon\nu\right)\int_0^t I_p(\rho)(s)\,\mathrm{d}s.\end{aligned}$$

Therefore, for $\varepsilon > 0$ small enough, $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^1_k)$ by Gronwall's inequality. We emphasize that this estimate also applies to the p-heat equation, i.e. (7.2) with $\lambda = 0$. Finally coming back to (7.5) yields

$$\begin{aligned}\left(1-\lambda C_{d,p}^{p-3}M_0^{3-p}\right)\int_0^t I_p(\rho)(s)\,\mathrm{d}s &\leq \int_{\mathbb{R}^d}\rho^{\mathrm{in}}\log\rho^{\mathrm{in}} - \int_{\mathbb{R}^d}\rho\log\rho\\ &\leq \int_{\mathbb{R}^d}\rho^{\mathrm{in}}\log\rho^{\mathrm{in}} + \nu_k\int_{\mathbb{R}^d}\rho m - M_0\ln M_0,\end{aligned}$$

and we conclude the proof using inequality (7.3).

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RÉSUMÉ

On étudie dans cette thèse le comportement en temps long de solutions d'équations aux dérivées partielles. Celles-ci modélisent des systèmes à grand nombre de particules dont la dynamique est due à des forces externes, internes et à l'interaction entre ces particules. Cependant, on considère différentes échelles. On voyage ainsi du niveau quantique des atomes au niveau macroscopique des étoiles, et l'on voit que des différences apparaissent bien que certaines propriétés soient conservées. Dans ce voyage, on croise le chemin de diverses applications telles que l'astrophysique, les plasmas, les semi-conducteurs, la biologie et l'économie. Ce travail est divisé en trois parties.

Dans la première, on étudie le comportement semi-classique de l'équation de Hartree en mécanique quantique et sa limite vers l'équation de Vlasov. On quantifie uniformément en la constante de Planck des propriétés telles que la propagation des moments et de normes de Lebesgue à poids et la dispersion. On les utilise ensuite pour établir des estimées de stabilité entre les deux équations au moyen d'un analogue semi-classique des distances de Wasserstein.

Dans la deuxième partie, on regarde le comportement en temps long d'équations cinétiques dont l'opérateur de collision est linéaire et a un équilibre local avec peu de moments, tel que l'opérateur de Fokker-Planck, sa version fractionnaire et un opérateur de Boltzmann linéaire. Deux principales techniques sont utilisées, l'une consistant à construire des entropies et la seconde à utiliser la positivité.

Enfin, la dernière partie s'intéresse à des modèles macroscopiques inspirés de l'équation de Keller-Segel et l'on regarde les paramètres sous lesquels ce type de système s'effondre sur lui-même, se disperse ou se stabilise. Le premier effet se voit en introduisant des poids appropriés, le deuxième avec des distances de Wasserstein et le troisième au moyen des normes de Lebesgue.

MOTS CLÉS

Systèmes dynamiques, particules en interaction, modèles cinétiques, mécanique quantique, Laplacien fractionnaire

ABSTRACT

In this thesis, we study the behavior of solutions of partial differential equations that arise from the modeling of systems with a large number of particles. The dynamic of all these systems is driven by interaction between the particles and external and internal forces. However, we will consider different scales and travel from the quantum level of atoms to the macroscopic level of stars. We will see that differences emerge from the associated dynamics even though the main properties are conserved. In this journey, we will cross the path of various applications of these equations such as astrophysics, plasma, semi-conductors, biology, economy. This work is divided in three parts.

In the first one, we study the semiclassical behavior of the quantum Hartree equation and its limit to the kinetic Vlasov equation. Properties such as the propagation of moments and weighted Lebesgue norms and dispersive estimates are quantified uniformly in the Planck constant and used to establish stability estimates in a semiclassical analogue of the Wasserstein distance between the solutions of these two equations.

In the second part, we investigate the long time behavior of macroscopic and kinetic models where the collision operator is linear and has a heavy-tailed local equilibrium, such as the Fokker-Planck operator, the fractional Laplacian with a drift or a Linear Boltzmann operator. This let appear two main techniques, the entropy method and the positivity method.

In the third part, we are interested in macroscopic models inspired from the Keller-Segel equation, and we study the range of parameters under which the system collapses, disperses or stabilizes. The first effect is studied using appropriate weights, the second using Wasserstein distances and the third using Lebesgue norms.

KEYWORDS

Dynamical systems, interacting particles, kinetic models, quantum mechanics, fractional Laplacian